ORDER-ENRICHED CATEGORICAL MODELS OF THE CLASSICAL SEQUENT CALCULUS

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ABSTRACT. It is well-known that weakening and contraction cause naïve categorical models of the classical sequent calculus to collapse to Boolean lattices. Starting from a convenient formulation of the well-known categorical semantics of linear classical sequent proofs, we give models of weakening and contraction that do not collapse. Cut-reduction is interpreted by a partial order between morphisms. Our models make no commitment to any translation of classical logic into intuitionistic logic and distinguish non-deterministic choices of cut-elimination. We show soundness and completeness via initial models built from proof nets, and describe models built from sets and relations.

1. INTRODUCTION

While the proof theory of propositional intuitionistic logic with disjunction, conjunction, and implication obtains a clean interpretation in bi-cartesian closed categories, it is well-known that adding the interpretation of a dualizing negation, to interpret the proof theory of propositional classical logic, makes the categories collapse to Boolean algebras [15, 14].

Classical natural deduction [19] may be represented as terms of the $\lambda\mu\nu$-calculus [17, 20]. Models of $\lambda\mu\nu$ can be obtained in fibrations over a base category of structural maps in which each fibre is a model of intuitionistic natural deduction and in which dualizing negation is interpreted as certain maps between the fibres [16, 20]. (Alternative models are given by control categories and co-control categories [22].) Whilst these solutions provides non-trivial categorical models, with computationally significant examples, it relies on a choice of $\neg\neg$-translations of classical logic into intuitionistic logic [24, 18]. Such a choice imposes a restriction on the equational theory of proofs which is most readily apparent when one considers cut-elimination in the classical sequent calculus [9]. To see this, consider the following example, due to Lafont [25, 12], in which the cut-redex has two possible reducts:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\Phi_1}{\vdash A}}{\vdash A, B}\,\mathrm{WR} \qquad \dfrac{\dfrac{\vdots\,\Phi_2}{\vdash A}}{B \vdash A}\,\mathrm{WL}}{\vdash A, A}\,Cut}{\vdash A}\,\mathrm{CR} \quad \preccurlyeq \quad \begin{array}{c}\vdots\,\Phi_1\\ \vdash A\end{array} \quad \text{or} \quad \begin{array}{c}\vdots\,\Phi_2\\ \vdash A\end{array}$$

The loss of the symmetry of the sequent calculus forced by $\lambda\mu\nu$'s choice of fibred model, admits only the reduction to Φ_2. In functional programming jargon, $\neg\neg$-translations are called "continuation-passing-style" (CPS) transforms [18], and the transform chosen above validates equalities (between $\lambda\mu\nu$ terms) typical for call-by-name. A call-by-value CPS transform would admit only the reduction to Φ_1. If the denotations of Φ_1 and Φ_2 are made equal, then the collapse of the categorical model follows.

Thus we seek a semantics of the classical sequent calculus which is both non-trivial (*i.e.*, not a Boolean algebra), and symmetric in the sense that there is no enforced commitment to a particular strategy of cut-elimination. To escape from the collapse, we shall weaken the assumption that the redex and the reduct of a cut-reduction must have the same denotation: we shall only require that the two be related by a *partial order* relation. Thus, we shall introduce a class of order-enriched categories to model the classical sequent calculus which are

(1) non-trivial in the sense that there are hom-spaces with more than one denotable element,
(2) sound in the sense that *all* cut-reductions are admitted by the partial order,
(3) complete in the same order-theoretic sense.

One challenge turns out to be the categorical interpretation of the structural rules. The naïve approach would be to use finite products (resp. coproducts) to interpret left (resp. right) weakening and contraction. But this would result in admitting both reductions in Lafont's example (in the sense that redex and reduct are equal), and so the models would collapse.

By contrast, it is known that there are non-trivial models of the *linear* fragment of the classical sequent calculus. To address the problems caused by the structural rules, we shall

(1) start with models of the linear fragment of the classical sequent calculus,
(2) endow every object A with a monoid $(\nabla : A \oplus A \longrightarrow A, [] : 0 \longrightarrow A)$ to model right contraction and weakening, and a co-monoid $(\Delta : A \longrightarrow A \otimes A, A \longrightarrow 1)$ to model left contraction and weakening,
(3) add an order-enrichment, and
(4) introduce some delicate conditions about the interaction between the monoids, the co-monoids, and the partial order.

Our chosen models of the linear fragment are *linearly distributive categories* [4] (formerly called "weakly distributive categories").

The resulting order-enriched categories will be sound and complete with respect to cut-reduction in the classical sequent calculus.

It is worth noting that, while our motivation is to present a non-trivial semantics of the classical sequent calculus, the redex in Lafont's example is actually intuitionistic, and contains neither negation nor implication. However, Lafont's example seems to rely crucially on the possibility of multiple succedents (*i.e.*, formulæ on the right side of the proof gate $\vdash$). Thus, the minimal setting for a semantic study of Lafont's example seems to be the *multi-succedent intuitionistic sequent calculus* [6] *without implication*. We implicitly cover this minimal setting, because our setting differs only in that we add negation *orthogonally*.

While sequent proofs are our conceptual starting point, they contain a good deal of extraneous information, which needlessly complicates the study of their semantics. This is well-known and one of the reasons why sequent calculi are studied via *proof nets*. Proof nets where introduced by Girard for studying linear logic [10]. A different kind of proof net was used in [2] to build initial linearly distributive categories. The connection between sequent proofs and proof nets is fairly obvious and has been repeatedly formulated in *sequentialization* theorems which state that every proof net can be turned into a sequent proof [10, 21] (the converse is almost trivial). Therefore, we shall switch from sequent proofs to proof nets early on in this article. The proof nets we use where introduced by Robinson [21] and possess rule nodes for weakening and contraction. They will in fact provide the initial categorical model from which we derive our completeness result.

1.1. **Construction of this article.** In § 2, we shall recall the definitions of the classical sequent calculus, Robinson's proof nets, and linearly distributive categories.

In § 3, we shall present cut-elimination for proof nets, thereby motivating § 4, where we introduce a notion of *net theory* with judgments of the form $M \preccurlyeq N$, which roughly mean that net M and be cut-reduced to net N.

In § 5, we explain how linearly distributive categories form a sound and complete semantics of *linear* proof nets (via an initial model build from proof nets). This is similar to [2], but there are some important differences.

In § 6, we shall show how to extend linearly distributive categories with monoids, co-monoids, and an order enrichment to provide a sound and complete semantics in the presence of weakening and contraction. The completeness proof will employ an initial model built from proof nets. This model will be unusually informative compared with typical term models in logics and computer science. The non-triviality of the semantics will follow from a simple model built from sets and relations.

2. Preliminaries

2.1. **Classical Sequent Calculus.** It is debatable what a natural-deduction system for classical logic should be, and none of the proposed systems (*e.g.*, [17]) adheres strictly to the introduction–elimination format. By contrast, the classical sequent calculus is quite definitive, and has remained remarkably stable since Gentzen. The main developments have been the investigation of tweaks to do with the placing of structural rules, and an understanding, inspired by Girard, of the different implications of choosing additive or multiplicative formulations of the rules. Later in this article, we shall introduce proof nets as a more economic notation of sequent proofs; one of the lessons there is that the theory is very smooth for the multiplicative connectives, but more problematic for the additives, which require "boxes" to indicate subproofs [11]. We therefore adopt a multiplicative presentation of classical logic.

A *sequent* has the form $\Gamma \vdash \Delta$, where both the *precedent* Γ and the *succedent* Δ are finite sequences of propositional logical formulæ as given by the grammar

$$A, B ::= A \wedge B \mid \top \mid A \vee B \mid \bot \mid \neg A \mid b$$

where b ranges over atomic formulæ. We consider implication to be derived — that is,

$$A \Rightarrow B \;:= \neg A \vee B$$

The inference rules are presented in Tables 1 and 2. It is helpful for our purposes to consider the left introduction rule $\top\mathrm{L}$ (which is missing in Table 1) as a degenerate case of Rule WL, with $A = \top$, and dually for $\bot\mathrm{R}$. When we refer to the classical sequent calculus, we mean the system presented in Tables 1 and 2. When we refer to the linear fragment of the classical sequent calculus, we mean the system presented in Table 1, plus the degenerate cases $\top\mathrm{L}$ and $\bot\mathrm{R}$ of the rules WL and WR, respectively.

2.2. **Linearly distributive categories.** Linearly distributive categories (formerly called "weakly distributive categories") where introduced by Seely and Cockett in [4]. They have two binary operations: a (tensor) "product" $\otimes$, and a "sum" $\oplus$. The key feature is a natural transformation

$$\delta : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$

called *linear distributivity* which is precisely what is needed to model Gentzen's cut rule (in the absence of other structural rules).

In this article, we shall only use *symmetric* linearly distributive categories, which have twist maps $A \otimes B \cong B \otimes A$ and $A \oplus B \cong B \oplus A$. This corresponds to the fact that the sequent calculus considered in this paper admits the exchange law.

Next, we turn towards the precise definition of a symmetric linearly distributive category. To help later reference, we shall present all details, starting with monoidal categories.

$$\frac{}{A \vdash A}\,\mathrm{Ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\,\wedge\mathrm{R}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,\vee\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}\,\vee\mathrm{L}$$

$$\frac{}{\vdash \top}\,\top\mathrm{R}$$

$$\frac{}{\bot \vdash}\,\bot\mathrm{L}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\,\neg\mathrm{L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\,\neg\mathrm{R}$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,\mathrm{EL} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,\mathrm{ER}$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\textsc{Cut}$$

TABLE 1. Linear inference rules of the Classical Sequent Calculus.

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{WL} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,\mathrm{WR}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{CL} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,\mathrm{CR}$$

TABLE 2. Weakening and contraction rules of the Classical Sequent Calculus.

A *monoidal category* is a category $\mathbf{C}$ together with a functor $\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$, an object 1, and natural isomorphisms

$$\alpha_\otimes : (A \otimes B) \otimes C \cong A \otimes (B \otimes C) \qquad \lambda_\otimes : 1 \otimes A \cong A \qquad \rho_\otimes : A \otimes 1 \cong A$$

satisfying the following coherence conditions.

(1)
$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_\otimes} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_\otimes} & A \otimes (B \otimes (C \otimes D)) \\ \downarrow{\scriptstyle \alpha_\otimes \otimes id} & & & & \uparrow{\scriptstyle id \otimes \alpha_\otimes} \\ (A \otimes (B \otimes C)) \otimes D & & \xrightarrow[\alpha_\otimes]{\qquad\qquad} & & A \otimes ((B \otimes C) \otimes D) \end{array}$$

(2)
$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{\alpha_\otimes} & A \otimes (1 \otimes B) \\ & {\scriptstyle \rho_\otimes \otimes id} \searrow \quad \swarrow {\scriptstyle id \otimes \lambda_\otimes} & \\ & A \otimes B & \end{array}$$

(3)
$$1 \otimes 1 \overset{\lambda_\otimes}{\underset{\rho_\otimes}{\rightrightarrows}} 1$$

A *symmetric monoidal category* is a monoidal category with a natural transformation $\sigma_\otimes : A \otimes B \cong B \otimes A$ such that

(4)
$$\begin{array}{ccc} A \otimes B & \xrightarrow{\sigma_\otimes} & B \otimes A \\ & {\scriptstyle id} \searrow & \downarrow{\scriptstyle \sigma_\otimes} \\ & & A \otimes B \end{array}$$

(5)
$$\begin{array}{ccc} 1 \otimes A & \xrightarrow{\sigma_\otimes} & A \otimes 1 \\ & {\scriptstyle \lambda_\otimes} \searrow \quad \swarrow {\scriptstyle \rho_\otimes} & \\ & A & \end{array}$$

(6)
$$\begin{array}{ccccc} (C \otimes A) \otimes B & \xrightarrow{\sigma_\otimes \otimes id} & (A \otimes C) \otimes B & \xrightarrow{\alpha_\otimes} & A \otimes (C \otimes B) \\ \downarrow{\scriptstyle \alpha_\otimes} & & & & \downarrow{\scriptstyle id \otimes \sigma} \\ C \otimes (A \otimes B) & \xrightarrow[\sigma_\otimes]{} & (A \otimes B) \otimes C & \xrightarrow[\alpha_\otimes]{} & A \otimes (B \otimes C) \end{array}$$

A *symmetric linearly distributive category* is a category $\mathbf{C}$ together with two symmetric monoidal structures $(\otimes, 1, \alpha_\otimes, \lambda_\otimes, \rho_\otimes, \sigma_\otimes)$ and $(\oplus, 0, \alpha_\oplus, \lambda_\oplus, \rho_\oplus, \sigma_\oplus)$ and a natural transformation $\delta : A \otimes$

$(B \oplus C) \longrightarrow (A \otimes B) \oplus C$ satisfying various coherence conditions. Before stating them, we define natural transformations δ^L_L, δ^L_R, δ^R_R, and δ^R_L as follows:

$$
\begin{array}{ccccccc}
A \otimes (B \oplus C) & \xrightarrow{id \otimes \sigma_\oplus} & A \otimes (C \oplus B) & \xrightarrow{\sigma_\otimes} & (C \oplus B) \otimes A & \xrightarrow{\sigma_\oplus \otimes id} & (B \oplus C) \otimes A \\
\delta \downarrow = \downarrow \delta^L_L & = & \downarrow \delta^L_R & = & \downarrow \delta^R_R & = & \downarrow \delta^R_L \\
(A \otimes B) \oplus C & \xrightarrow[\sigma_\oplus]{} & C \oplus (A \otimes B) & \xrightarrow[id \oplus \sigma_\otimes]{} & C \oplus (B \otimes A) & \xrightarrow[\sigma_\oplus]{} & (B \otimes A) \oplus C
\end{array}
$$

In our statement of the coherence conditions, we shall use the following three symmetries (taken from [4]):

op'**:** Reverse the arrows and swap $\otimes$ and $\oplus$, as well as 1 and 0. This gives the following assignment of maps:

$$
\begin{array}{ccccccccc}
\delta^L_L & \leftrightarrow & \delta^R_R & \alpha_\otimes & \mapsto & \alpha_\oplus^{-1} & \alpha_\oplus & \mapsto & \alpha_\otimes^{-1} \\
\delta^L_R & \mapsto & \delta^L_R & \rho_\otimes & \mapsto & \rho_\oplus^{-1} & \rho_\oplus & \mapsto & \rho_\otimes^{-1} \\
\delta^R_L & \mapsto & \delta^R_L & \lambda_\otimes & \mapsto & \lambda_\oplus^{-1} & \lambda_\oplus & \mapsto & \lambda_\otimes^{-1} \\
 & & & \sigma_\otimes & \mapsto & \sigma_\oplus^{-1} & \sigma_\oplus & \mapsto & \sigma_\otimes^{-1}
\end{array}
$$

$\otimes'$**:** Reverse the tensor $\otimes$; this assigns

$$
\begin{array}{ccccccccc}
\delta^L_L & \leftrightarrow & id^R_L & \alpha_\otimes & \mapsto & \alpha_\otimes^{-1} & \alpha_\oplus & \mapsto & \alpha_\oplus \\
\delta^L_R & \leftrightarrow & \delta^R_R & \rho_\otimes & \leftrightarrow & \lambda_\otimes & \rho_\oplus & \mapsto & \rho_\oplus \\
 & & & & & & \lambda_\oplus & \mapsto & \lambda_\oplus \\
 & & & \sigma_\otimes & \mapsto & \sigma_\otimes^{-1} & \sigma_\oplus & \mapsto & \sigma_\oplus
\end{array}
$$

$\oplus'$**:** Reverse the tensor $\oplus$; this assigns

$$
\begin{array}{ccccccccc}
\delta^L_L & \leftrightarrow & \delta^L_R & \alpha_\otimes & \mapsto & \alpha_\otimes & \alpha_\oplus & \mapsto & \alpha_\oplus^{-1} \\
\delta^R_L & \leftrightarrow & \delta^R_R & \rho_\otimes & \mapsto & \rho_\otimes & \rho_\oplus & \leftrightarrow & \lambda_\oplus \\
 & & & \lambda_\otimes & \mapsto & \lambda_\otimes & & & \\
 & & & \sigma_\otimes & \mapsto & \sigma_\otimes & \sigma_\oplus & \mapsto & \sigma_\oplus^{-1}
\end{array}
$$

The coherence laws are as follows, where for each law we also require all versions generated by the symmetries op', $\otimes'$, and $\oplus'$:

(7)
$$
\begin{array}{ccc}
1 \otimes (A \oplus B) & & \\
\delta^L_L \downarrow & \searrow^{\lambda_\otimes} & \\
(1 \otimes A) \oplus B & \xrightarrow[\lambda_\otimes \oplus id]{} & A \oplus B
\end{array}
$$

$$\begin{array}{ccc} (A\otimes B)\otimes(C\oplus D) & \xrightarrow{\alpha_\otimes} & A\otimes(B\otimes(C\oplus D)) \\ & & \downarrow{\scriptstyle id\otimes\delta^L_L} \\ \downarrow{\scriptstyle \delta^L_L} & & A\otimes((B\otimes C)\oplus D) \\ & & \downarrow{\scriptstyle\delta^L_L} \\ ((A\otimes B)\otimes C)\oplus D & \xrightarrow[\alpha_\otimes\oplus id]{} & (A\otimes(B\otimes C))\oplus D \end{array} \tag{8}$$

$$\begin{array}{ccc} & (A\oplus B)\otimes(C\oplus D) & \\ {\scriptstyle\delta^L_L}\swarrow & & \searrow{\scriptstyle\delta^R_R} \\ ((A\oplus B)\otimes C)\oplus D & & A\oplus(B\otimes(C\oplus D)) \\ \downarrow{\scriptstyle\delta^R_R\oplus id} & & \downarrow{\scriptstyle id\oplus\delta^L_L} \\ (A\oplus(B\otimes C))\oplus D & \xrightarrow[\alpha_\oplus]{} & A\oplus((B\otimes C)\oplus D) \end{array} \tag{9}$$

$$\begin{array}{ccc} A\otimes((B\oplus C)\oplus D) & \xrightarrow{id\otimes\alpha_\oplus} & A\otimes(B\oplus(C\oplus D)) \\ \downarrow{\scriptstyle\delta^L_L} & & \downarrow{\scriptstyle\delta^L_R} \\ (A\otimes(B\oplus C))\oplus D & & B\oplus(A\otimes(C\oplus D)) \\ \downarrow{\scriptstyle\delta^L_R\oplus id} & & \downarrow{\scriptstyle id\oplus\delta^L_L} \\ (B\oplus(A\otimes C))\oplus D & \xrightarrow[\alpha_\oplus]{} & B\oplus((A\otimes C)\oplus D) \end{array} \tag{10}$$

For further discussion of the structure of symmetric linearly distributive categories, see [4]. For the sake of brevity, we shall write "linearly distributive category" instead of "symmetric linearly distributive category" from here on.

To see how the linear distributivity can be used to model the cut rule, let $f : A \longrightarrow B\oplus C$ and $g : C\otimes D \longrightarrow E$ be morphisms. Then the *cut of f and g with cut object C* is

$$cut(f,g) := A\otimes D \xrightarrow{f\otimes id} (B\oplus C)\otimes D \xrightarrow{\delta^R_R} B\oplus(C\otimes D) \xrightarrow{B\oplus g} B\oplus E$$

A *linearly distributive category with negation* is a linearly distributive category together with, for every object A, an object A^*, and maps

$$\gamma^L : A^*\otimes A \longrightarrow 0 \qquad\qquad \tau^R : 1 \longrightarrow A\oplus A^*$$

Together with the induced maps

$$\gamma^R : A\otimes A^* \longrightarrow 0 \qquad\qquad \tau^L : 1 \longrightarrow A^*\oplus A$$

the following coherence conditions are required:

$$A \otimes 1 \xrightarrow{id \otimes \tau^L} A \otimes (A^* \oplus A) \xrightarrow{\delta^L_L} (A \otimes A^*) \oplus A \xrightarrow{\gamma^R \oplus id} 0 \oplus A, \qquad A \otimes 1 \xrightarrow{\rho_\otimes} A, \qquad 0 \oplus A \xrightarrow{\lambda_\oplus} A \tag{11}$$

$$A^* \otimes 1 \xrightarrow{id \otimes \tau^R} A^* \otimes (A \oplus A^*) \xrightarrow{\delta^L_L} (A^* \otimes A) \oplus A^* \xrightarrow{\gamma^L \oplus id} 0 \oplus A^*, \qquad A^* \otimes 1 \xrightarrow{\rho_\otimes} A^*, \qquad 0 \oplus A^* \xrightarrow{\lambda_\oplus} A^* \tag{12}$$

For further discussion of the structure of (symmetric) linearly distributive categories with negation, see [4].

Remark 1. As we shall see, linearly distributive with negation provide a sound and complete semantics of the linear fragment of the classical sequent calculus. There is, in fact, an alternative class of models whose definition does not require a linear distributivity, because it can be derived from universal property of negation. These alternative models, which the authors introduced as "bi-$*$-autonomous categories" [3], are based on two families of adjunctions

$$\frac{A \otimes B \longrightarrow C}{A \longrightarrow B^* \oplus C} \qquad\qquad \frac{A \longrightarrow B \oplus C}{A \otimes B^* \longrightarrow C}$$

These enable the derivation of a linear distributivity:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \oplus B \xrightarrow{f} A \oplus B}{(A \oplus B) \otimes A^* \longrightarrow B} \qquad \cfrac{B \otimes C \xrightarrow{g} B \otimes C}{B \longrightarrow C^* \oplus (B \otimes C)}}{(A \oplus B) \otimes A^* \longrightarrow C^* \otimes (B \otimes C)}}{((A \oplus B) \otimes A^*) \otimes C \longrightarrow B \otimes C}}{((A \oplus B) \otimes C) \otimes A^* \longrightarrow B \otimes C}}{(A \oplus B) \otimes C \longrightarrow A \oplus (B \otimes C)}$$

It can be shown that bi-$*$-autonomous categories (whose definition contains quite a few coherence conditions not mentioned above) are equivalent to linearly distributive categories.

Bi-$*$-autonomous categories seem quite appealing owing to their clear explanation of negation. However, linearly distributive categories are ultimately much easier to work with, which is why we finally adopted them as the basis of our semantics.

2.3. **Proof nets.** Proof nets were introduced by Girard for the study of linear logic [10]. They have been applied to various other logical systems [2, 1]. In this article, they play a key rôle in the semantic analysis of the Classical Sequent Calculus. The proof nets we use are the two-sided sequent-style nets for classical logic recently introduced by Robinson [21].

Roughly speaking, a proof net is a connected graph built from the figures in Tables 3 and 4, satisfying a certain global condition. We begin formalizing this by recalling the following definition from [21].

Definition 1. A *proof structure* is a bipartite directional graph whose two families of nodes are labeled as follows:

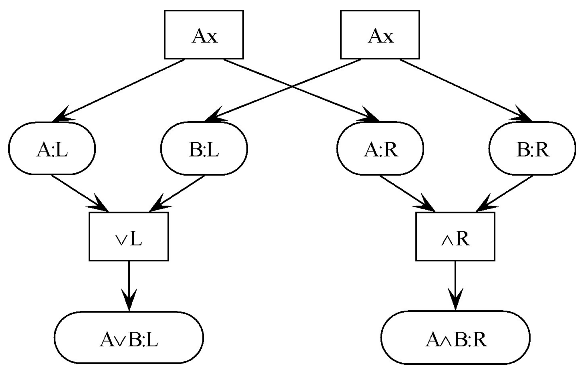
Family 1: labeled by one of the sequent proof rules;
Family 2: labeled by a formula, together with the information Left or Right.

The graph is subject to the following additional constraints:

(1) The graph surrounding each rule node is given uniquely as an instance of the corresponding figure (in Table 3 or 4);
(2) Each propositional node has a unique incoming and at most one outgoing arc.

There is some ambiguity in the phrase "the graph surrounding each rule node is given uniquely as an instance of the corresponding figure". We intend that this mapping is given as part of the structure of the graph. In most instances, only one such mapping will be possible, but we will wish to distinguish the two inputs to, say, an $\wedge\mathrm{R}$ even when they are instances of the same formula.

However, we still have structures which do no represent valid proofs, for example



These structures are eliminated by using a technique due to Danos and Regnier [5].

Definition 2. A *(Danos-Regnier) switching* σ is the choice of one of the hypotheses for each node of the following forms: $[\wedge\mathrm{L}]$, $[\vee\mathrm{R}]$, $[\mathrm{CL}]$, $[\mathrm{CR}]$. We shall say that the remaining nodes are *unswitched*.

The purpose of a switching is to generate a graph.

Definition 3. Let S be a proof structure and σ a switching on it. Then the (Danos-Regnier) graph of σ, $DR(\sigma, S)$, is the following undirected graph:

- Its vertices are the propositional vertices of S;
- Its edges join conclusions of rule nodes to hypotheses as follows. If the rule node is unswitched, then each conclusion is joined to each hypothesis. If the rule node is switched, then the conclusion is joined only to the hypothesis chosen by σ. The exceptions are axioms and cut, where the two formulæ are joined.

Definition 4. A proof structure S is a *proof net* if for each switching σ of S the Danos-Regnier graph of σ, $DR(\sigma, S)$, is connected and acyclic (as an undirected graph).

There is a straightforward procedure to turn a sequent proof into a proof net (see [21]). However, the converse is a substantial theorem.

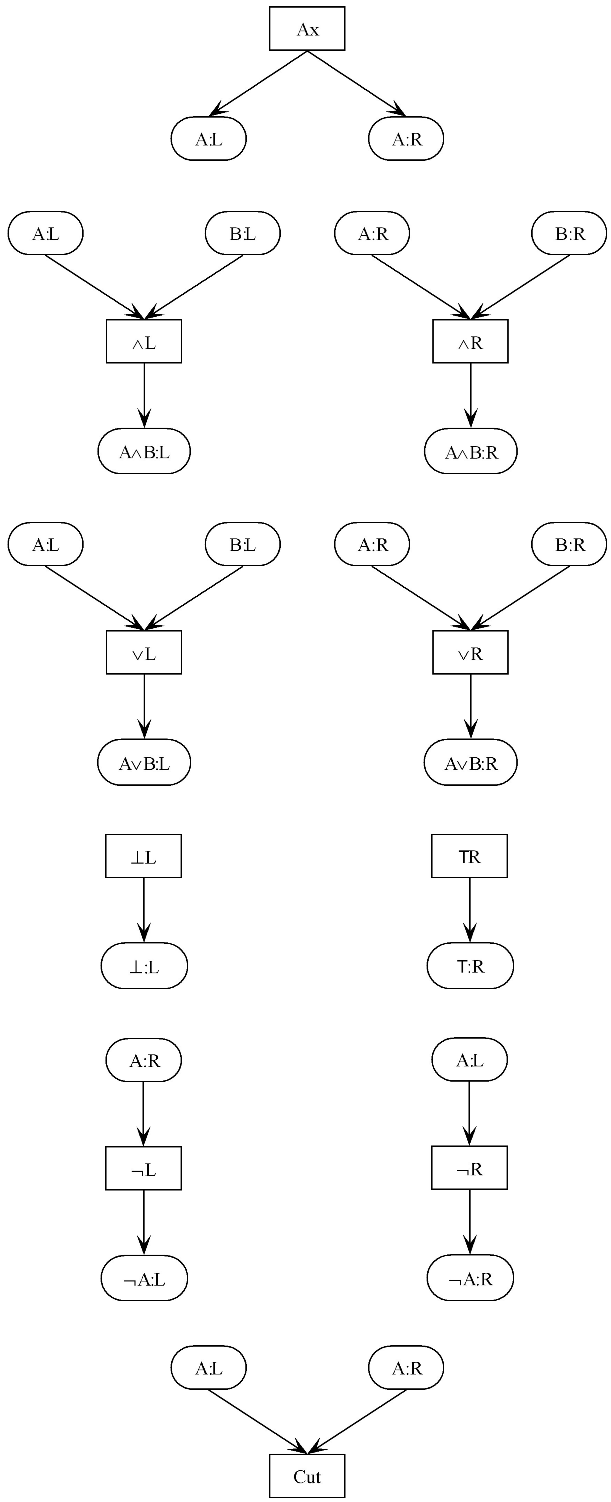


TABLE 3. Proof nets: linear rules

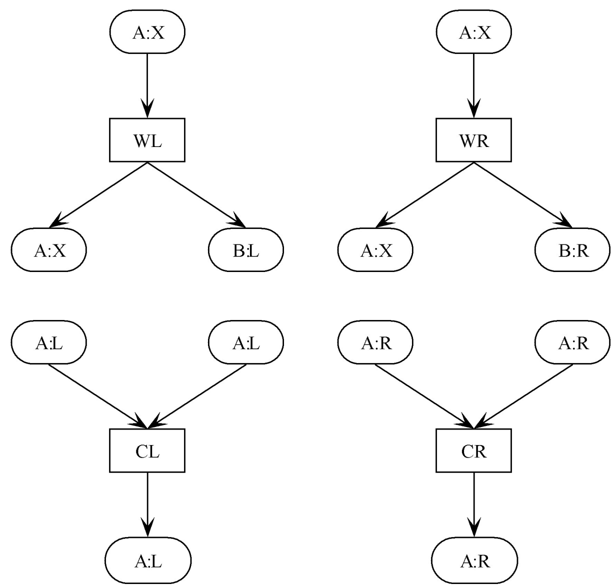
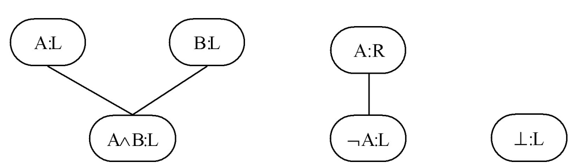


TABLE 4. Proof nets: structural rules

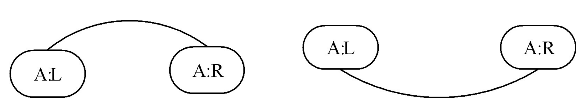
Theorem 2.1. *Every proof net can be generated as the image of a sequent proof.*

For the proof nets we introduced above, this theorem has been proved by Robinson [21]. (However, for different kinds of proof nets, such theorems have been proved before.)

One possible reading of the sequentialization theorem is that a proof structure is a proof net if and only if it can be built from the figures in Tables 3 and 4 *inductively* like a sequent proof. Before we formalize this, we introduce a more economic notation for nets, which is obtained as follows: First, adopt the convention that proof structures are drawn in such a way that all of their edges point downwards. Second, omit the arrowheads, which are now redundant. Third, for figures other than AX and CUT, remove the rule nodes and connect the hypotheses directly with the conclusions. For example, the figures for $\wedge$L, $\neg$L, and $\perp$L are represented by

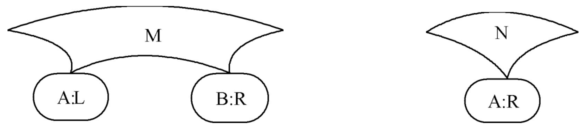


respectively. The figures for AX and CUT are represented by



respectively. Note that by shifting to the new notation we loose no information.

Now for the inductive presentation of proof nets. We let figures like



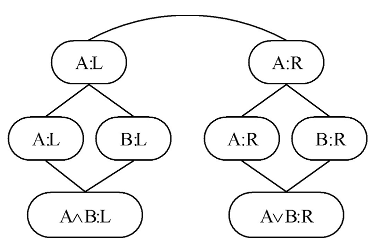
range over proof nets; in this example, M has doors $A : L$ and $B : R$ (and maybe more), and N has a door $A : R$ (and maybe more). The inductive definition of proof nets is presented in Tables 5 and 6. We call a net *linear* if it contains no rule nodes of the form CL, CR, WL, or WR, with the exception that WL (resp. WR) is allowed if the formula introduced by the weakening is $\top$ (resp. $\bot$), in which case we call the rule $\top\mathrm{L}$ (resp. $\bot\mathrm{R}$).

A propositional node of a net is called a *door* if it has no outgoing arcs. The *kingdom* (resp. *empire*) of a propositional node A is the smallest (resp. largest) subnet with A as a door. It is proved in [21] that the kingdom and empire of a propositional node always exist.

As mentioned in the introduction, sequent proofs contain extraneous information which is discarded in proof nets. For example, consider the sequent proof

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\,\mathrm{Ax}}{A, B \vdash A}\,\mathrm{WL}}{A, B \vdash A, B}\,\mathrm{WR}}{A \wedge B \vdash A, B}\,\wedge\mathrm{L}}{A \wedge B \vdash A \vee B}\,\vee\mathrm{R}$$

There are six variations of this proof with respect to the order in which the inference rules are used: (1) WL-WR-$\wedge\mathrm{L}$-$\vee\mathrm{R}$, (2) WL-WR-$\vee\mathrm{R}$-$\wedge\mathrm{L}$, (3) WL-$\wedge\mathrm{L}$-WR-$\vee\mathrm{R}$, (4) WR-WL-$\wedge\mathrm{R}$-$\vee\mathrm{L}$, (5) WR-WL-$\vee\mathrm{L}$-$\wedge\mathrm{R}$, (6) WR-$\wedge\mathrm{R}$-WL-$\vee\mathrm{L}$. The proof net corresponding to this proof, and all of its variations, is



This illustrates that, by using proof nets, we no longer have to deal with permutations of rules. In fact, the suppression of permutations is the only information loss in the transition from sequent proofs to proof nets. (For a precise statement, see Proposition 3 in [21].) However, it greatly simplifies our presentation.

Remark 2. In this article, we shall build linearly distributive categories from linear proof nets (*i.e.*, proof nets without weakening and contraction), as a first step towards our semantics of the classical sequent calculus. In [2] too, proof nets are used in the construction of linearly distributive categories. However, there are important differences between the nets in [2] and the ones we use. Our choice of net is motivated by the study of cut-reduction. To this end, we need explicit cut links. In [2], where the main purpose is showing categorical coherence, explicit cut links are not present, and not needed.

We shall in fact present, in loving detail, a cut-elimination procedure for nets. In our linearly distributive category, composition will be defined in terms of the cut rule (not simply juxtaposition,

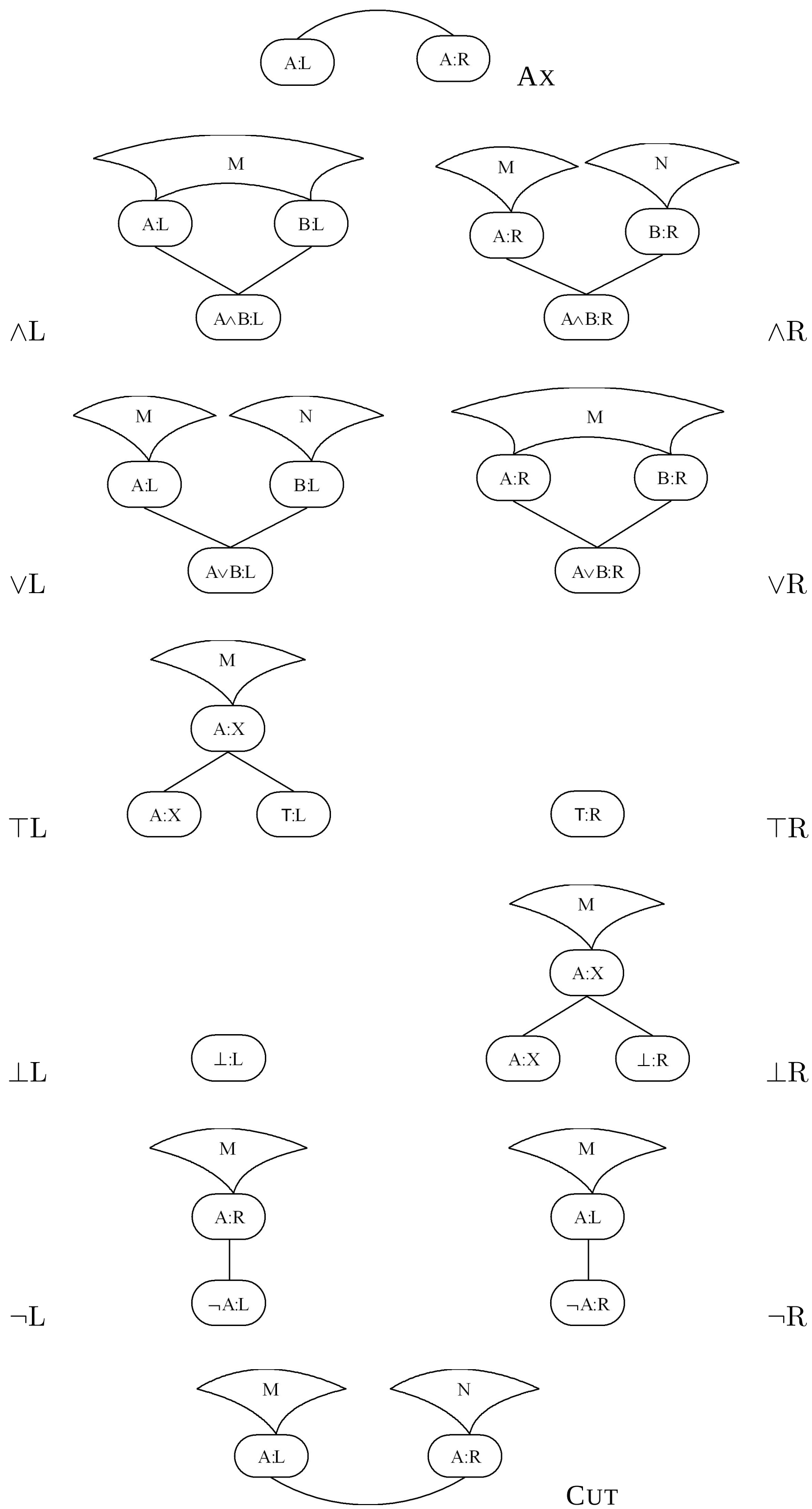


TABLE 5. Inductive definition of proof nets: linear rules

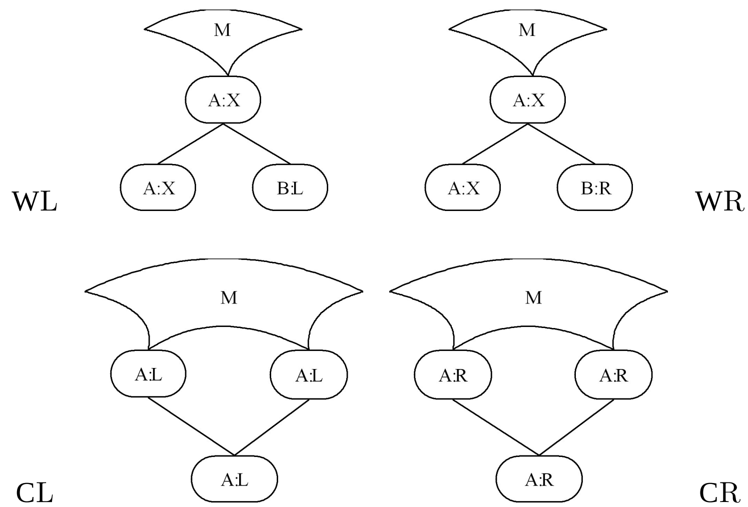


TABLE 6. Inductive definition of proof nets: structural rules

as in [2]). The categorical identities will be axioms (not the empty net, as in [2]). A good illustration of the differences between both kinds of nets is our cut-reduction law CUTAX in Table 7. It is a well-known step in cut-elimination, yet in cannot be stated in terms of the nets used in [2] (where it is trivially valid, though).

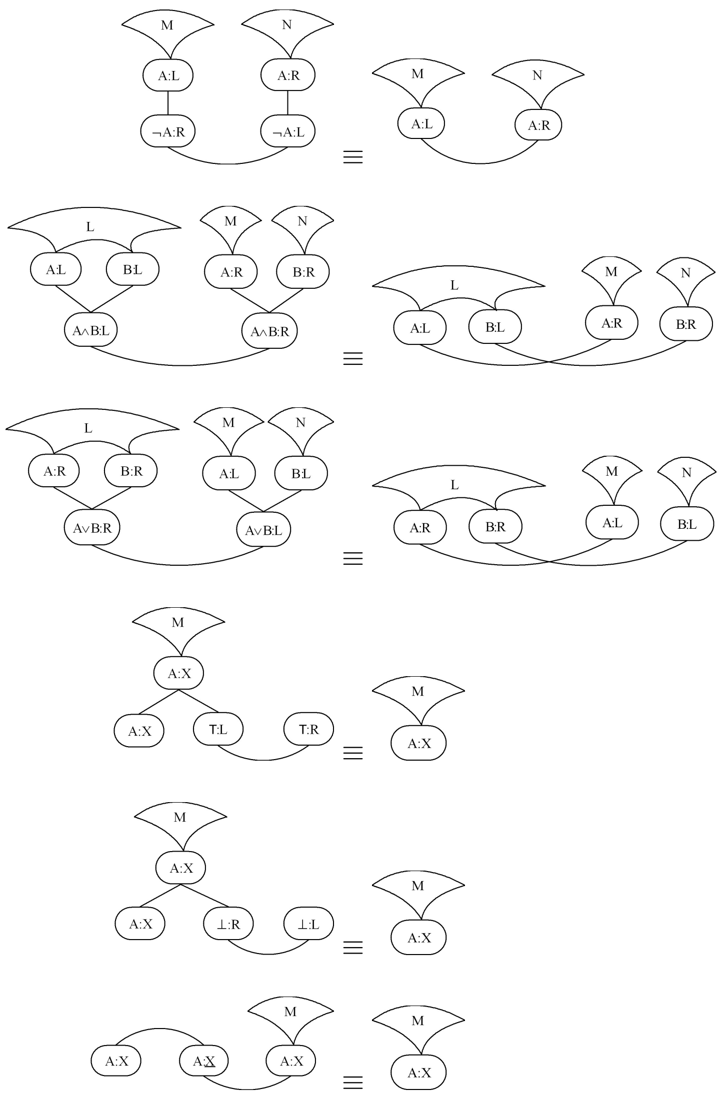
Also, the way in which we present the equality of nets differs from that in [2]. For example, there is a striking difference in the axiomatization of the units $\top$ and $\bot$. (In fact, our axiomatization requires fewer equational laws than the one in [2]. This is possible because we allow ourselves to use the *non-local* law W-MOVE.) In particular, our *empire re-wiring* result (Proposition 3.4), which is essentially the same as Proposition 3.3 in [2], is proved in a very different way.

3. CUT-ELIMINATION FOR NETS

Proof nets are our chosen representation of classical proofs, for which we are seeking a sound and complete semantics. It is therefore essential to have a precise definition of equality between nets. This equality must be based on cut-reduction, because that is the phenomenon we want to model. We shall therefore present a cut-elimination procedure for Robinson's nets [21], to demonstrate that the spirit of our investigation does not depart from the sequent calculus. Our starting point is Robinson's discussion of cut-reduction [21], from which cut-elimination is obtained (essentially) following Gentzen [9] in the usual way. (Robinson has also worked independently on cut-elimination.)

The cut-reduction rules presented in this section will form the basis of our definition of *net theories* in $\S$ 4.

The rules we use for cut-elimination are presented in Tables 7, 8, and 9. We use $M \equiv N$ as an abbreviation for the pair of rules $M \preccurlyeq N$ and $N \preccurlyeq M$. The capital letters X, Y, and Z range over L and R. We define a notation for switching sides, $\underline{L} = R$ and $\underline{R} = L$, which is used in Rules CUTAX, CUTW, and CUTC to avoid having to write two versions of each rule. The Rules CUT$\neg$, CUT$\wedge$,



CUT¬

CUT∧

CUT∨

CUT⊤

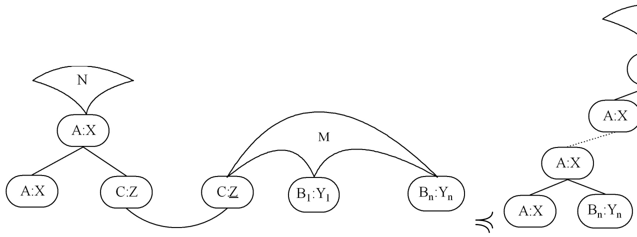
CUT⊥

CUTAX

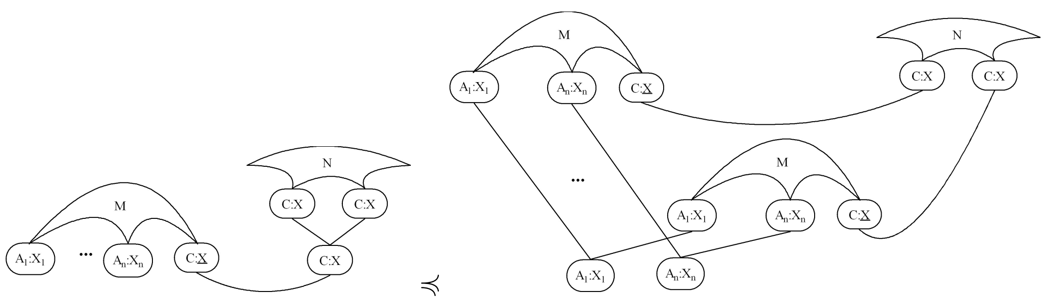
TABLE 7. Local cut-reductions: logical cuts and cuts against an axiom

CUT∨, CUT⊤, and CUT⊥ are the well-known reductions of "logical cuts". Rules CUTAX, CUTW, and CUTC are also well known. The importance of the rules in Table 9 will become evident in the cut-elimination proof.

The presentation of cut-reduction on nets highlights an aspect which is not so evident when sequent proofs are used: the cut-reduction rules in Table 7 are *local* in the sense that only a tiny subgraph of the net is rewritten. (The same is true for the coherence laws in Table 9.) By contrast, the rules CUTW and CUTC are non-local: the changes may copy or discard arbitrarily large parts of the net.



CUTW



CUTC

TABLE 8. Non-local cut-reductions: cuts against weakening and contraction

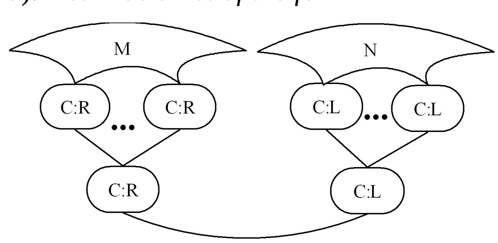
Therefore, we call the rules in Table 7 the *local cut-reductions*, and the rules in Table 8 the *non-local cut-reductions*.

(Note that we use $\equiv$ in all rules except in the non-local cut-reductions, However, in the cut-elimination proof we shall use the rules CUT$\wedge$, CUT$\vee$, CUT$\neg$, and CUTAX only from left to right. We shall justify the use of $\equiv$ in $\S$ 4.)

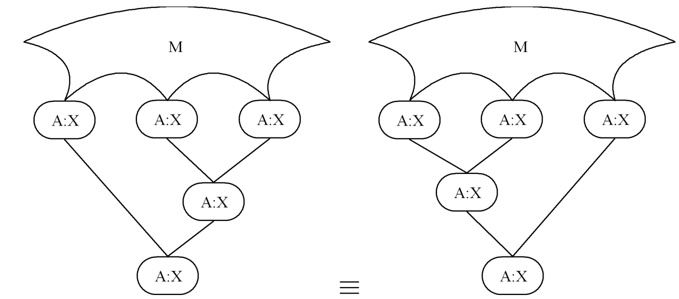
Lemma 3.1 (Coherence of contraction)**.** *Let M be a net with $n+1$ doors of the form $A : L$. Let M_1 and M_2 be any two nets that result from M by n applications of* CL *to the $A : L$ (so finally only one $A : L$ is left). Then M_1 and M_2 are equivalent modulo* C-ASSOC*,* C-CROSS*, and* C-TWIST*. Dually for doors of the form $A : R$.*

Now follows the Principal Lemma for cut-elimination. The horizontal dots stand for multiple contractions (whose arrangement does not matter by Lemma 3.1).

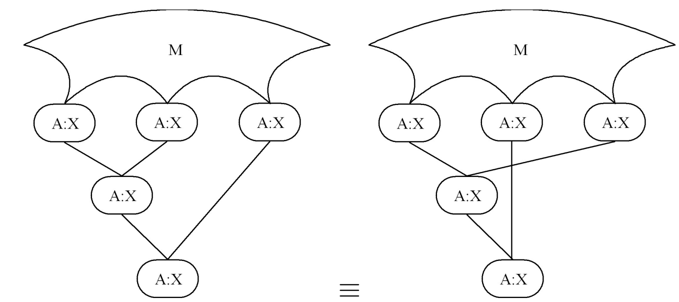
Lemma 3.2 (Principal Lemma)**.** *Let L be a net of the form*



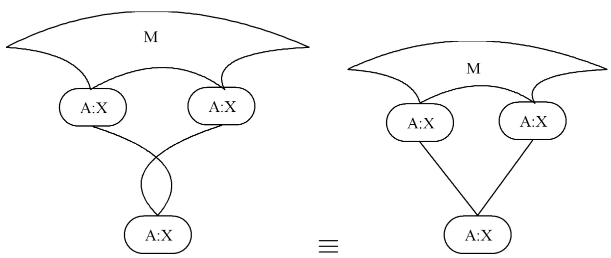
where M and N are cut-free. Then L can be transformed into a cut-free net by using the rules in Tables 7, 8, and 9.



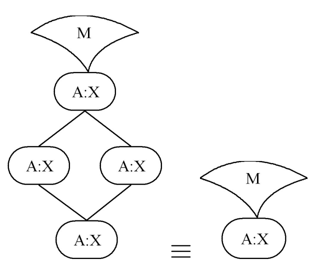
C-ASSOC



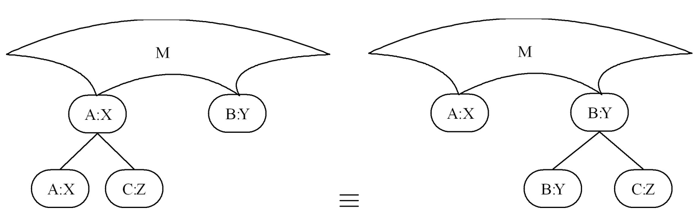
C-CROSS



C-TWIST



WC



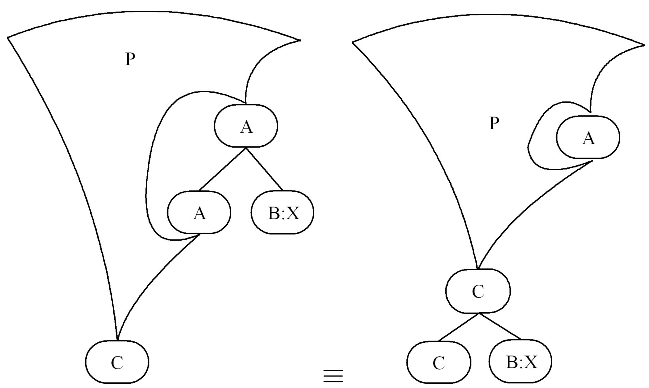
W-MOVE

TABLE 9. Coherence laws needed for cut-elimination

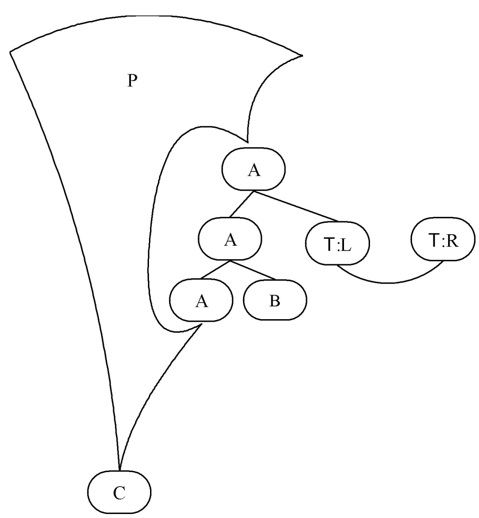
Thus, we essentially use *multicuts*, as in Gentzen's original proof.

Before we can prove the Principal Lemma, we need to prove a crucial *re-wiring proposition* (Prop. 3.4), which states essentially that weakening links can be moved around freely. The re-wiring proposition is necessary because, in contrast to the sequent calculus, weakenings in nets must be attached to some existing node. Our re-wiring proposition is similar to Proposition 3.3 in [2]. However, as mentioned in Remark 2, our axiomatization of the equivalence $\equiv$ of nets differs from that in [2]. Therefore, a new re-wiring proof is in order. First, a lemma:

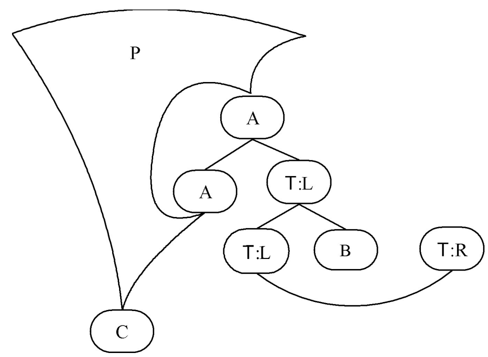
Lemma 3.3. *Whenever one (and therefore both) sides below are nets, one side can be transformed into the other by using rules* CUT$\top$ *(alternatively,* CUT$\bot$*) and* W-MOVE.



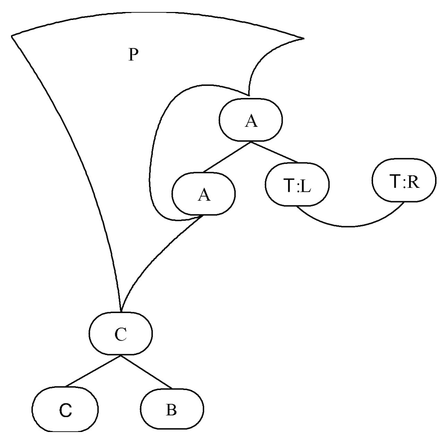
Proof. First, we transform the left net into the right one. By applying Rule CUT$\top$ backwards to the kingdom (or empire) of the upper A, we obtain



By applying Rule W-MOVE to the kingdom of B, we obtain



By applying Rule W-MOVE to the empire of the lower $\top : L$, we obtain



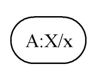
Now the right net in the statement of the lemma results from applying Rule CUT$\top$ forwards to the empire of the lower A. All rules we used are reversible, so we can also obtain the left net from the right one. $\square$

Proposition 3.4 (Empire re-wiring)**.** *Weakenings can be moved around freely within the empire of the formula they introduce, by using rules* CUT$\top$ *(alternatively,* CUT$\bot$*) and* W-MOVE*.*

Proof. By using Lemma 3.3 first forwards and then backwards. $\square$

Our proof of the Principal Lemma relies on notions of *rank* of a proof and *degree* of a cut, and proceeds by induction on the measure $(degree, rank)$, ordered lexicographically.

To define the net-version of the notion of *rank*, we allow doors to be "marked". A marked door has the form



where $x \in \{0, 1\}$. If $x = 1$, we call the door *marked*, otherwise *unmarked*. Next, we define a *decomposition relation* $\Rightarrow$ between marked nets in Table 10. Intuitively, we have $M \Rightarrow M'$ if M' is an immediate subnet of M. However, the key property of $\Rightarrow$ is the propagation of marks along doors: marks are propagated along contractions and along one side of weakenings, but not along introduction rules. The *left rank* $\mathrm{rank}_L(M)$ of a net M with at least one marked left door is the maximum length n of a sequence $M = M_1 \Rightarrow M_2 \Rightarrow \cdots \Rightarrow M_n$ such that all M_i have at least one marked left door. The *right rank* $\mathrm{rank}_R(M)$ of a net M with at least on marked right door is the maximum length n of

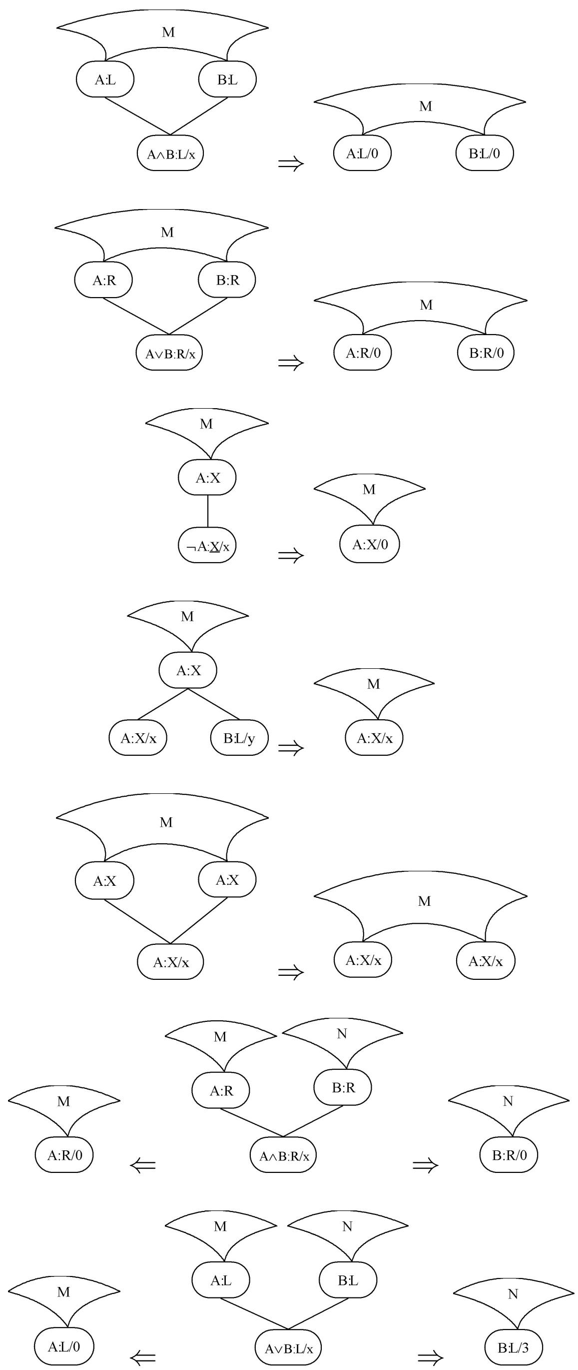
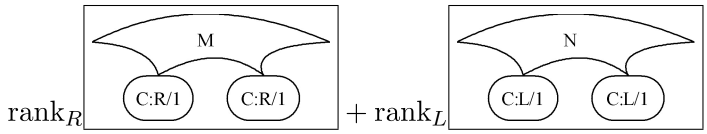


TABLE 10. Decomposition relation for determining left and right rank of marked nets

a sequence $M = M_1 \Rightarrow M_2 \Rightarrow \cdots \Rightarrow M_n$ such that all M_i have at least one marked right door. The *rank* of a multicut as required in the Principal Lemma is determined by the following marking:

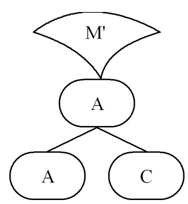


The *degree* of a cut is defined to be the number of logical operators contained in the cut formula.

Proof of the Principal Lemma. By induction over the measure $(degree, rank)$ of L, ordered lexicographically.

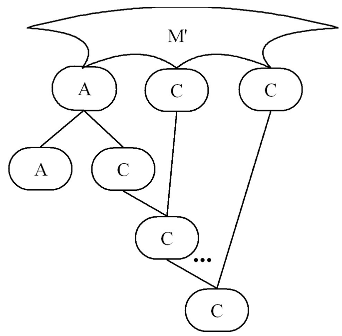
In the case in which $\mathrm{rank}_R(M) = \mathrm{rank}_L(N) = 1$, we proceed by a induction over the degree of the cut. In this case, degree is reduced but rank increases, so illustrating the need for the lexicographical ordering of $(degree, rank)$.

If M or N is an axiom, the cut can be eliminated by Rule CUTAX. If M, say, ends with a weakening, then (because of its minimal right rank) it must be of the form

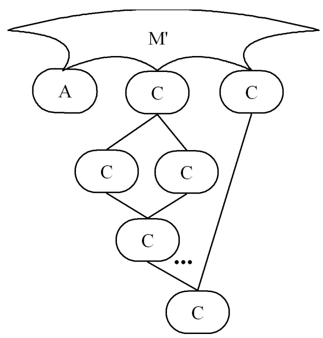


where C is the only marked door. Therefore, the cut can be eliminated by Rule CUTW. Dually for the case where N ends with a weakening. Owing to their minimal ranks, neither M nor N can end with a contraction. This leaves the case where both M and N end with the introduction rule of a connective. For reasons of rank, it must hold for both M and N that the introduced formula is the only marked C. In particular, M and N introduce the same connective. So the cut is a logical cut, and one of the rules in Table 7 applies. That rule produces cuts of lower degree, and those can be eliminated, by induction.

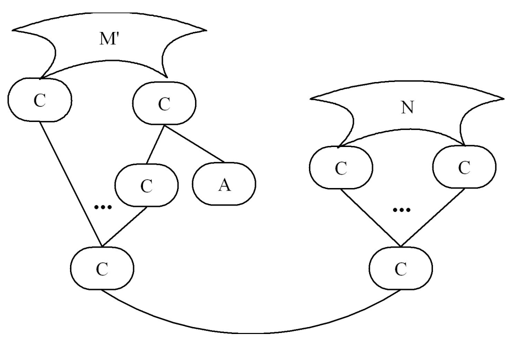
Now for the case where $\mathrm{rank}_R(M) + \mathrm{rank}_L(N) > 2$, in which degree is fixed and we argue by reduction of rank. Without loss of generality, suppose that $\mathrm{rank}_R(M) > 1$. Because of its non-minimal right rank, M cannot be an axiom. Suppose M ends with a contraction. If the door of the contraction is not among the marked Cs, then the contraction can be removed, and the cut can be eliminated by induction hypothesis. Otherwise, we can remove the mark of the contraction's conclusion and mark the contraction's hypotheses instead ("shrinking M by expanding the multi-contraction"). After that, the cut can be eliminated by induction hypothesis. Now suppose that M ends with a weakening. There are four sub-cases. (1) If neither of the weakening's conclusions is among the marked Cs, then the weakening can be removed, and the cut can be eliminated by induction hypothesis. (2) If both of the weakening's conclusions are among the marked Cs, then we can apply Rule WC, which enables the induction hypothesis. Now suppose that exactly one of the weakening's conclusions is among the marked Cs. (3) If that conclusion is the formula newly introduced by the weakening, then M's side of the multicut, modulo coherence of contraction, looks as follows:



By Rule W-MOVE, this is equivalent to



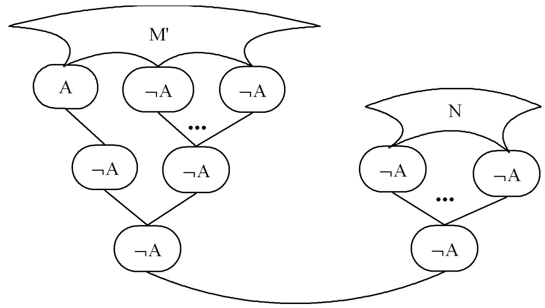
Now we can apply Rule WC, which enables the induction hypothesis. (4) If the marked conclusion of the weakening is the formula to which the weakening was introduced, we have the situation below:



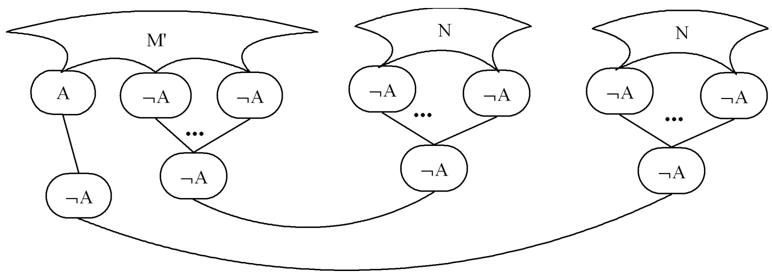
Either M' has a door B other than the two Cs, or N has a door B other than its marked Cs, for otherwise the removal of the weakening would yield a proof of the empty sequent, in contradiction to the system's evident logical consistency. Owing to the re-wiring proposition, our rewrite rules allow us to move the weakening to B. After that, the weakening can be removed, and the cut can be eliminated by induction hypothesis.

What remains is the case where M ends with a (right) introduction rule. Because $\mathrm{rank}_R(M) > 1$, that rule cannot be $\top\mathrm{R}$. We already covered the case $\bot\mathrm{R}$, because it is a special form of weakening. Now suppose the last rule of M is $\neg\mathrm{R}$, resulting in a door $\neg A$. If that door is not among the marked Cs, the negation-introduction can be removed, and the cut can be eliminated by the induction hypothesis.

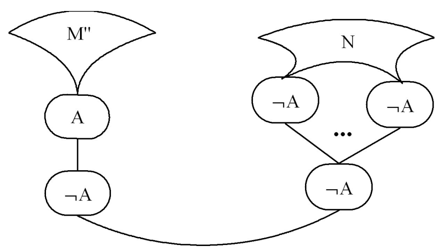
Otherwise, we have the situation below.



Using CUTC, we can transform this into the following net L' (where for the sake of presentation, we omit drawing the outer contractions joining the two copies of N):



The empire of A (*i.e.*, the net which is the multicut between M' and the left of the two N) satisfies the induction hypothesis, so the multicut can be eliminated, resulting in some cut-free net M''. Thus, we obtain a net



The key point now is that we can assume without of loss of generality that the right rank of this multicut is 1. For if this is not so, we can remove parts of M'' until it becomes true (see Table 10). Thus, the cut can eliminated by induction hypothesis.

The case where M ends with $\vee$R works in the same way.

Now for the case where M ends with $\wedge$R. It is similar to the cases for $\neg$R and $\vee$R, except for some minor complications: let $A \wedge B$ be the conclusion of that final $\wedge$R. Then M consists of a net M_A with A as a door and a net M_B with B as a door, linked by the final $\wedge$R. There are two subcases. (1) The conclusion of the final $\wedge$R is among the marked Cs. (a) If both M_A and M_B have a door $A \wedge B$ among the marked Cs, then two applications of CUTC (creating three copies of N) yield three multicuts; the multicut involving M_A and the multicut involving M_B have smaller rank than the original multicut, and can therefore be eliminated by induction hypothesis. For the remaining multicut, we can assume without loss of generality that its left rank is 1 (for reasons similar to the negation case explained above). So the third cut too can be eliminated by induction hypothesis. (b) If only one of M_A and M_B have a door $A \wedge B$ among the marked Cs, we need only one application of CUTC, but from then on the argument is the same as for (a). (2) Suppose the conclusion of the final $\wedge$R is not among the marked Cs. If only M_A, say, has a door $A \wedge B$ among the marked Cs, then the induction hypothesis applies in a trivial way. If both M_A and M_B have doors among the marked Cs, then one application of CUTC (creating two copies of N) yields two cuts to which the induction hypothesis applies, and they can be eliminated independently. □

The cut-elimination theorem follows immediately from the Principal Lemma:

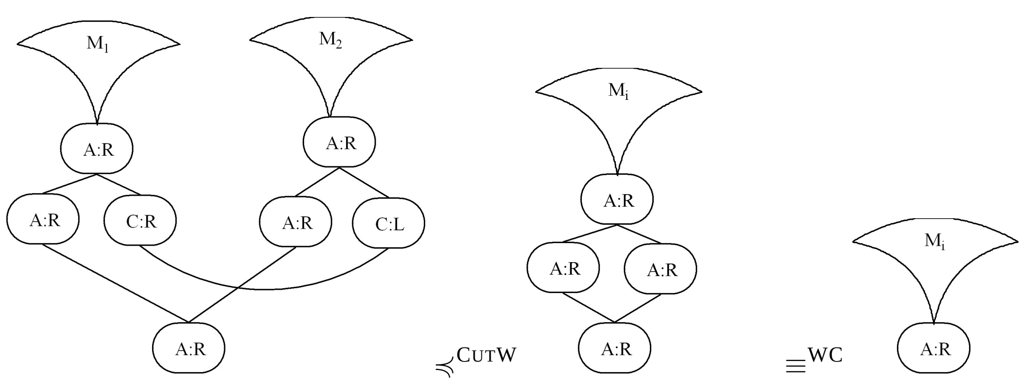
Theorem 3.5. *Every net can be transformed into a cut-free one by using the rules in Tables 7, 8, and 9.*

4. Net theories

Having studied cut-elimination, we are now ready to define the notions of equality and inequality between proofs.

Linearly distributive categories provide the standard categorical semantics of the linear fragment of the classical sequent calculus, and they admit all linear cut-reductions. They have nothing to say (and do not need to say anything) about non-symmetric judgments $M \preccurlyeq N$: either the morphisms denoted by nets M and N are equal or not. The case where the denotations are equal corresponds to our judgments $M \equiv N$. We want to keep this standard semantics of the linear fragment. The local cut-reductions (Table 7) take place in the linear fragment, which we want to keep modeling by linearly distributive structure. Therefore, we require these reductions to be *invertible*—that is, we require redex and reduct to be related via $\equiv$, which is $\preccurlyeq \cap \succcurlyeq$.

The non-local cut-reductions (Table 8) cannot be kept as equalities because they rule out interesting models: in the introduction, we already mentioned Lafont's example, which shows that requiring CUTW to be invertible rules out *all* interesting models. The net-version of Lafont's example looks as follows:

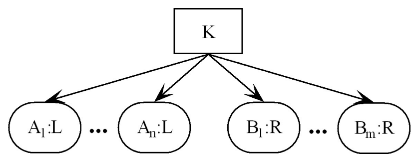


So if CUTW was invertible, then $M_1 \equiv M_2$, and therefore all proofs of A would be equivalent. One could of course blame WC instead of CUTW, but this would be evidently contrived, because WC is a coherence law whose absence would cause the structure to be an abomination. By contrast, we shall see in § 6 that dropping the invertibility of CUTW can be achieved by softening the naturality of projections into a lax naturality, which is a well-established categorical concept.

The invertibility of CUTC also kills important models. While we have no evidence that it makes any two proofs of a formula A equivalent, we shall see in § 6 that it rules out a desirable model: the category $\mathbf{Rel}$ of sets an relations. As we shall see, dropping the invertibility of CUTC can be achieved by softening the naturality of diagonals into a lax naturality, whereby $\mathbf{Rel}$ becomes a model.

In this section, we shall define a notion of *net theory* whose judgments are inequalities of the form $M \preccurlyeq N$, where M and N are nets (with matching sequences of doors). This notion of theory consists essentially of the local cut-reductions (invertible), the non-local cut-reductions (not invertible), the coherence laws presented in Table 9 (which we motivated by cut-elimination), and some more coherence laws explained in this section.

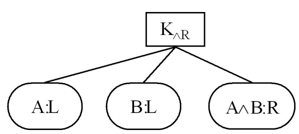
A *signature with negation* Σ consists of a set $\mathcal{A}_\Sigma$ of atomic formulæ and a set $\mathcal{K}_\Sigma$ of *constant nodes*

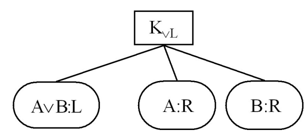


with at least one door, where the formulæ in the doors are generated from $\mathcal{A}_\Sigma$. Similarly, we define the notion of *net-signature*, the only difference being that the formulæ in the doors must be negation-free.

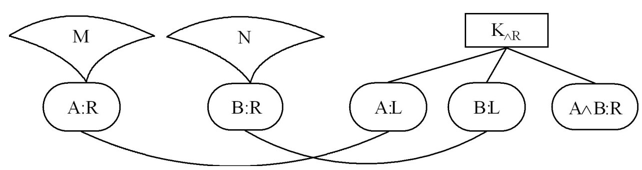
Remark 3. Constant nodes can cause logical inconsistency, like constants in the λ-calculus (*e.g.*, a fixpoint constant of type $(A \to A) \to A$). Also, constant nodes can evidently obstruct cut-elimination. But obviously, they are needed if nets are to serve as an "internal language" of the categories.

Constant nodes allow an important technical improvement: when we introduce constant nodes

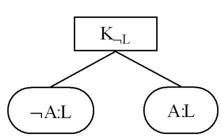


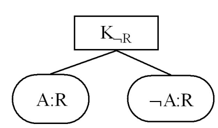


then the figure for $\wedge\mathrm{R}$ in Table 5 can be seen as an abbreviation for



and dually for $\vee\mathrm{L}$. This anticipates the categorical semantics we shall present. (The benefits of using these two constants were pointed out in [4], where they were used, under the names m_{AB} and w_{AB}, in the definition of "two-tensor polycategories".) In fact, constant nodes can be used to replace all unswitched rules (except CUT). In particular, we can replace $\neg\mathrm{L}$ and $\neg\mathrm{R}$ by constants





(However, we shall not introduce constants for WL and WR, because they bring no advantage.)

Remark 4. There seems to be an analogy with the lambda-calculus: its higher-order nature allows to add extra structure as constants (*e.g.*, $\mathit{fix} : (A \to A) \to A$ or $\mathit{case} : (A + B) \to (A \to C) \to (B \to C) \to C$). Analogously, the cut rule of the sequent calculus allows to add unswitched rules as constants.

A *net over a signature* Σ *with negation* is a graph generated from elements of $\mathcal{K}_\Sigma$ according to Definition 4, except that the rules $\vee\mathrm{L}$, $\wedge\mathrm{R}$, $\neg\mathrm{L}$, and $\neg\mathrm{R}$ are replaced by the respective constant nodes. Also, from here on, we assume a linear order on the left doors, and a linear order on the right doors.

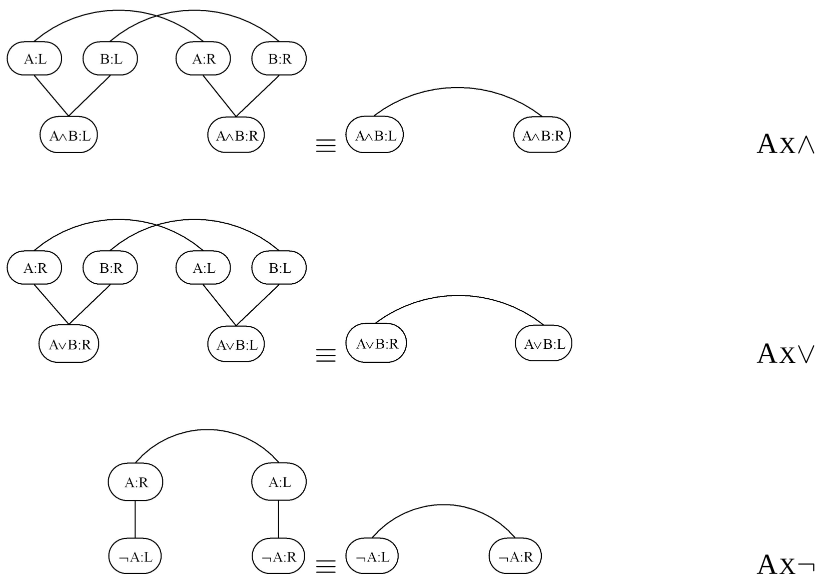


TABLE 11. Expansions of axioms

Definition 5. A *net theory* $\mathcal{T}$ over Σ is a set of inequalities $M \preccurlyeq N$ where M and N are nets over Σ (with matching sequences of doors), with the following properties:

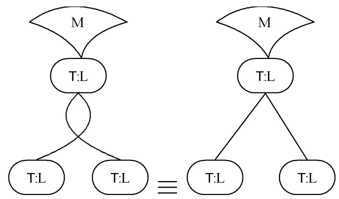
(1) The relation $\preccurlyeq$ is reflexive, transitive, and compatible (*i.e.*, all net-formation rules are "monotonic" with respect to $\preccurlyeq$);
(2) The rules in Tables 7–12 hold (where $M \equiv N$ means $M \preccurlyeq N$ and $N \preccurlyeq M$).

The equality laws in Tables 11 and 12 are easy to justify: the axiom expansions in Table 11 are widely used by logicians. In the category we shall construct from nets, they correspond to the laws $id_A \otimes id_B = id_{A \otimes B}, id_A \oplus id_B = id_{A \oplus B}$, and $(id_A)^* = id_{A^*}$. The equation TWIST$\top$ is an evident coherence law: it states that if we introduce a $\top$ on the left when there already is another $\top$, we cannot distinguish the two afterwards. Dually for TWIST$\bot$. The laws W$\wedge$ and C$\wedge$ state that the rules $\wedge$L, WL, and CL interact in a coherent way. Dually for the laws W$\vee$ and C$\vee$. As we shall see in § 6, the laws in Table 12, together with those in Table 9, amount to requiring that the category has *monoids* and *co-monoids*.

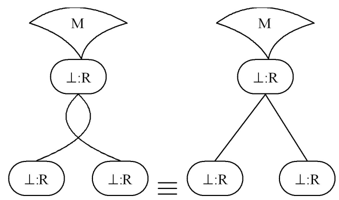
5. LINEAR NETS AND LINEARLY DISTRIBUTIVE CATEGORIES

In this section, we shall show that introduce *linear-net theories* are in perfect correspondence with linearly distributive categories. Linear-net theories have neither structural rules nor negation, and the only kind of judgment is of the form $M \equiv N$. Much of our analysis reconstructs that which is found in [2] but does so for Robinson's nets [21], which are directly based on the sequent calculus. It is necessary for our subsequent development.

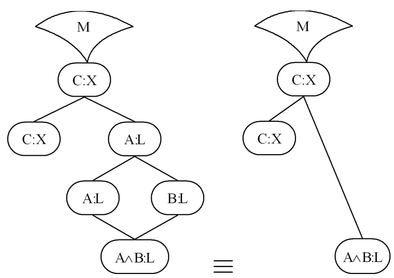
In § 5.1, we shall present the interpretation of linear-net theories in linearly distributive categories. The ambiguity of the decomposition of proof nets (*i.e.*, the fact that it cannot generally be determined



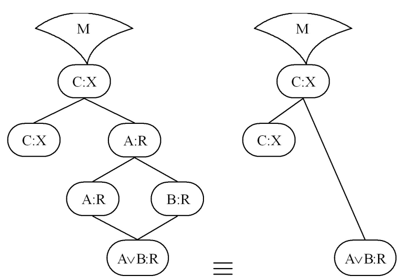
TWIST⊤



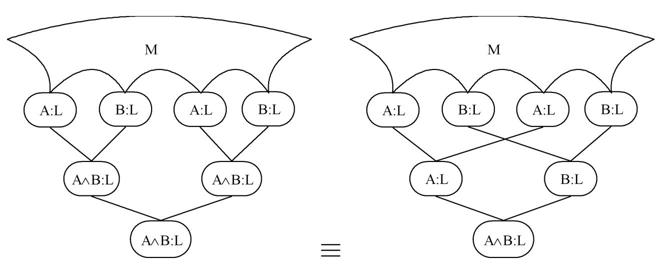
TWIST⊥



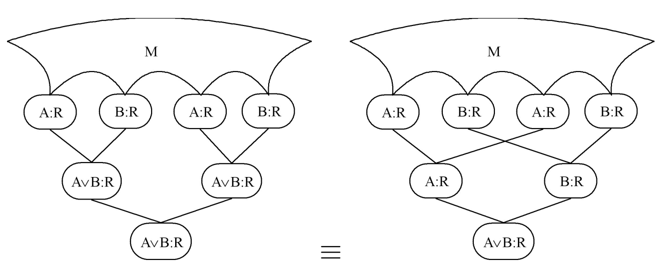
W∧



W∨



C∧



C∨

TABLE 12. Remaining coherence laws for net theories

which rule is the "last") necessitates a proof that the inductively-defined interpretation is well-defined. This corresponds to the fact that the very syntax of proof nets already encodes some equalities of linearly distributive categories. We shall make this precise in Theorem 5.1.

In § 5.2, we shall show that every linear-net theory forms a linearly distributive category (Theorem 5.4) which is an initial model (Theorem 5.8), and prove completeness (Theorem 5.7).

In § 5.3, we shall add negation and show that all previous results carry over without problems.

As explained in Remark 2, a correspondence between proof nets and linearly distributive categories has already been shown [2], but the nets we use, and our definition of equality between them, differ from the ones in [2] because of our focus on cut-reduction. Therefore, we need to discuss this correspondence in detail.

Definition 6. A *linear net over a signature* Σ is a net over Σ without occurrences of $K_{\neg\mathrm{L}}$, $K_{\neg\mathrm{R}}$, CL and CR, such that WL occurs only with $B = \top$ (in which case we write $\top\mathrm{L}$ instead of WL), and WR occurs only with $B = \bot$ (in which case we write $\bot\mathrm{R}$ instead of WL).

Definition 7. A *linear-net theory* $\mathcal{T}$ over a signature Σ is a set of equalities $M \equiv N$ where M and N are linear nets over Σ (with matching sequences of doors), such that $\equiv$ is a congruence which contains all instances of CUT$\wedge$, CUT$\vee$, CUT$\top$, CUT$\bot$, CUTAX, W-MOVE for $C \in \{\bot, \top\}$, AX$\wedge$, AX$\vee$, TWIST$\top$, and TWIST$\bot$.

So linear-net theories consist of equational judgments $M \equiv N$, in contrast to net-theories, which consist of inequational judgments $M \preccurlyeq N$. (However, the right conceptual view is that linear-net theories have judgments $M \preccurlyeq N$ where $\preccurlyeq$ happens to be symmetric.)

5.1. **Categorical interpretation of linear nets.** An *interpretation* of a linear-net theory $\mathcal{T}$ in a linearly distributive category $\mathbf{C}$ sends a formula A of $\mathcal{T}$ to an object $\lfloor A \rfloor$ of $\mathbf{C}$ according to the rules

$$\lfloor A \wedge B \rfloor = \lfloor A \rfloor \otimes \lfloor B \rfloor \qquad \lfloor \top \rfloor = 1 \qquad \lfloor A \vee B \rfloor = \lfloor A \rfloor \oplus \lfloor B \rfloor \qquad \lfloor \bot \rfloor = 0$$

(So an interpretation of formulæ is determined by the interpretation atomic formulæ.) As mentioned earlier, the interpretation of nets cannot simply proceed by induction, because the ambiguity of decomposition. We shall therefore start with an interpretation of *serialized* linear nets, which are nets together with information that removes this ambiguity: whenever there are two or more potential "last" rules, the extra information specifies the choice of one rule. After defining the interpretation, we shall prove that it does not depend on the serialization (Theorem 5.1).

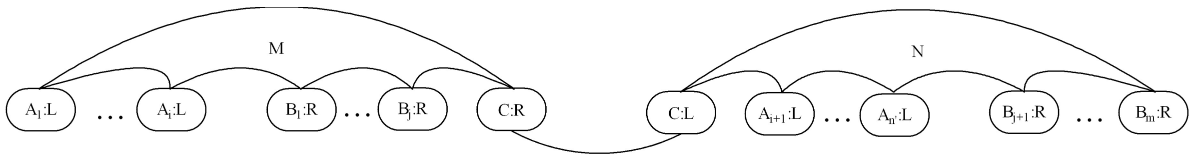
A serialized linear net with left doors $A_1, \ldots, A_n$ and right doors $B_1, \ldots, B_m$ is interpreted by a morphism

$$\lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor \longrightarrow \lfloor B_1 \rfloor \oplus \cdots \oplus \lfloor B_m \rfloor$$

where $\otimes$ and $\oplus$ are deemed to be, say, left associative, the nullary product is 1, and the nullary sum is 0. The rule AX is interpreted by the identity. The rules $\wedge$L and $\top$L are interpreted by pre-composing the corresponding symmetric monoidal isomorphism associated with $\otimes$. Dually, $\vee$R and $\bot$R are interpreted by post-composing the corresponding symmetric monoidal isomorphism associated with $\oplus$. The rule CUT is interpreted by the categorical operator cut, which takes as arguments two morphisms $f : A \longrightarrow B \oplus C$ and $g : C \otimes D \longrightarrow E$ and is defined as follows:

$$cut(f, g) := A \otimes D \xrightarrow{f \otimes id} (B \oplus C) \otimes D \xrightarrow{\delta^R_R} B \oplus (C \otimes D) \xrightarrow{B \oplus g} B \oplus E$$

Specifically, the interpretation of the net



is given as follows:

$$\dfrac{\dfrac{A_1 \otimes \cdots \otimes A_i \xrightarrow{\lfloor M \rfloor} B_1 \oplus \cdots \oplus B_j \oplus C}{A_1 \otimes \cdots \otimes A_i \longrightarrow (B_1 \oplus \cdots \oplus B_j) \oplus C}\,\text{sm}\oplus \qquad \dfrac{C \otimes A_{i+1} \otimes \cdots \otimes A_n \xrightarrow{\lfloor N \rfloor} B_{j+1} \oplus \cdots \oplus B_m}{C \otimes (A_{i+1} \otimes \cdots \otimes A_n) \longrightarrow B_{j+1} \oplus \cdots \oplus B_m}\,\text{sm}\otimes}{\dfrac{(A_1 \otimes \cdots \otimes A_i) \otimes (A_{i+1} \otimes \cdots \otimes A_n) \longrightarrow (B_1 \oplus \cdots \oplus B_j) \oplus (B_{j+1} \oplus \cdots \oplus B_m)}{A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)} \longrightarrow B_{\tau(1)} \oplus \cdots \oplus B_{\tau(m)}}\,\text{sm}\otimes,\text{sm}\oplus}\,\mathit{cut}$$

where sm$\otimes$ stands for pre-composing symmetric monoidal isomorphisms associated with $\otimes$, sm$\oplus$ stands for post-composing symmetric monoidal isomorphisms associated with $\oplus$, and σ and τ are the permutations corresponding to the order of the new net's doors. The constant nodes $K_{\wedge \mathrm{R}}$ and $K_{\vee \mathrm{L}}$ are interpreted by $id_{\lfloor A \rfloor \otimes \lfloor B \rfloor}$ and $id_{\lfloor A \rfloor \oplus \lfloor B \rfloor}$, respectively.

Evidently, an interpretation of serialized linear nets is determined by the interpretation of the *non-logical constant nodes* (*i.e.*, those which are not $K_{\vee \mathrm{L}}$ or $K_{\wedge \mathrm{R}}$).

Theorem 5.1. *For every interpretation $\lfloor - \rfloor$ of sequentialized linear nets, it holds that $\lfloor M' \rfloor = \lfloor M'' \rfloor$ whenever M' and M'' are serializations of the same linear net M.*

Proof. Let $\mathbf{C}$ be a linearly distributive category, let Σ be a signature, and let $\lfloor - \rfloor$ be an interpretation of the serialized linear nets over Σ in $\mathbf{C}$. For every linear net M, we have the *set* of morphisms

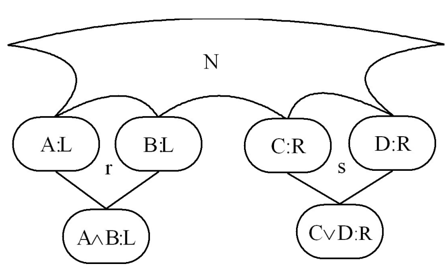
$$S(M) = \{\lfloor M' \rfloor : M' \text{ is a serialization of } M\}$$

We prove that, for all M, the set $S(M)$ has only one element. The proof proceeds by induction over the size of M. The base cases (Ax, $\bot$L, $\top$R, and constant nodes) are trivial: there is only one serialization of M. Now for the induction step. For every final rule r of M, define

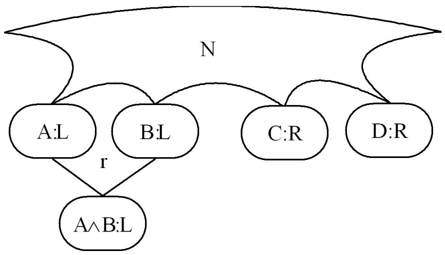
$$S_r(M) := \{\lfloor M' \rfloor : M' \text{ is a serialization of } M \text{ whose last rule is } r\}$$

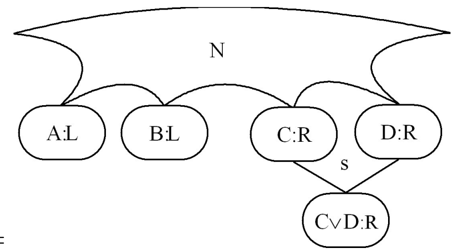
Obviously, $S(M)$ is the union of all the $S_r(M)$ (where r ranges over the final rules of M). By the induction hypothesis, all serializations of M minus r have the same interpretation. Therefore, every $S_r(M)$ is a singleton set. So it suffices to prove that for every two final rules r and s, the sets $S_r(M)$ and $S_s(M)$ are equal. We proceed by a case split on (r, s).

Case 1: To warm up, consider the case r is of type $\wedge$L and s is of type $\vee$R. Then M must have the form



Let g be the morphism which is (by induction hypothesis) the only element of $S(N)$. Define nets N_r and N_s as follows:

$N_r :=$ 

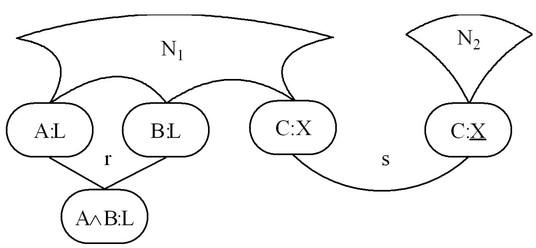
$N_s :=$ 

Let g_r (resp. g_s) be the morphism which is (by induction hypothesis) the only element of $S(N_r)$ (resp. $S(N_s)$). By our definition of interpretation, we have $g_r = g \circ i_r$, where i_r is the symmetric monoidal isomorphism associated with $\otimes$ that "puts the brackets around $\lfloor A \rfloor \otimes \lfloor B \rfloor$". Dually, $g_s = i_s \circ g$, where i_s is the symmetric monoidal isomorphism associated with $\oplus$ that "puts the brackets around $\lfloor C \rfloor \oplus \lfloor D \rfloor$". Now let M_r be a serialization of M with last rule r. Let f_r be the morphism which is the interpretation of M_r. By definition of our notion of interpretation, we have $f_r = g_s \circ i_r = (i_s \circ g) \circ i_r$. Dually, let M_s be a serialization of M with last rule s, and let f_s be the morphism which is the interpretation of M_s. We have $f_s = i_s \circ g_r = i_s \circ (g \circ i_r)$. By associativity of $\circ$, we have $f_r = f_s$. So $S_r(M) = S_s(M)$.

This case for $r = \wedge\mathrm{L}$ and $s = \vee\mathrm{R}$ has a straightforward generalization to the case where $r \in \{\wedge\mathrm{L}, \top\mathrm{L}\}$ and $s \in \{\vee\mathrm{R}, \bot\mathrm{R}\}$, because all that matters is that r is interpreted by pre-composing a morphism, and s is interpreted by post-composing a morphism. The categorical law which is finally used is the associativity of $\circ$.

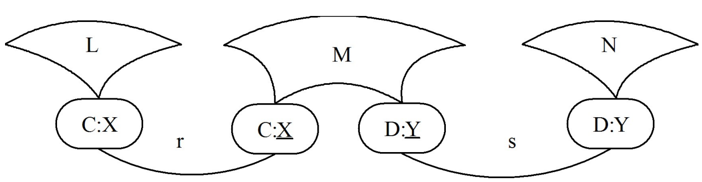
Case 2: $r, s \in \{\wedge\mathrm{L}, \top\mathrm{L}\}$. In this case, we end up in a situation where $f_r = g \circ i_s \circ i_r$ and $f_s = g \circ j_r \circ j_s$ where i_r, i_s, j_s, and j_r are symmetric monoidal isomorphisms associated with $\otimes$, and g is the interpretation of some subnet N. We get $f_r = f_s$ because, by symmetric monoidal coherence, $i_r \circ i_s = j_s \circ j_r$. Dually for the case $r, s \in \{\vee\mathrm{R}, \bot\mathrm{R}\}$.

Case 3: $r \in \{\wedge\mathrm{L}, \top\mathrm{L}, \vee\mathrm{R}, \bot\mathrm{R}\}$ and $s = \textsc{Cut}$. (All that matters about r here is that it is interpreted by pre- or post-composing a morphism.) Without loss of generality, let $r = \wedge\mathrm{L}$. The situation is as follows:



We get $f_r = f_s$ because pre-composition of a morphism commutes with the categorical operator cut, as can be easily checked.

Case 4: $r, s = \textsc{Cut}$. Then we have



The two possible cases, $X = Y$ and $X \neq Y$, correspond to two laws of *polycategories* which are well-known to hold in a linearly distributive category (Laws 3 and 4 in Definition 1.1 in [4].)

□

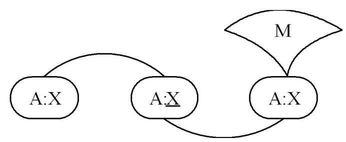
Now we turn towards the soundness proof. It relies heavily on the following lemma, which is the semantic counterpart of CUTAX.

Lemma 5.2. *The equation below holds in every linearly distributive category.*

$$\frac{U \xrightarrow{f} V \oplus A \qquad A \otimes 1 \xrightarrow{\rho_\otimes} A}{U \otimes 1 \longrightarrow V \oplus A}\,cut \quad = \quad U \otimes 1 \xrightarrow{\rho_\otimes} U \xrightarrow{f} V \oplus A$$

Proposition 5.3 (Soundness)**.** *Let $\lfloor - \rfloor$ be an interpretation of linear nets over some signature Σ. Then the judgments $M \equiv N$ such that $\lfloor M \rfloor = \lfloor N \rfloor$ form a linear-net theory.*

Proof. First we prove the soundness of CUTAX. Without loss of generality, suppose that $X = R$. If the domain and codomain of $\lfloor M \rfloor$ are U and W, respectively, then the interpretation of

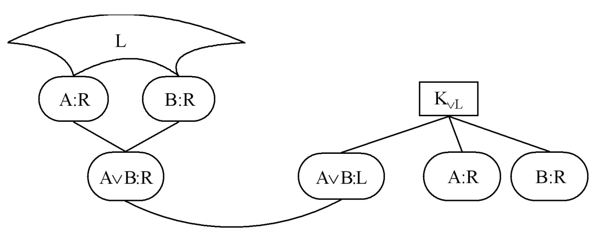


is

$$\dfrac{\dfrac{\dfrac{U \xrightarrow{\lfloor M \rfloor} W}{U \longrightarrow V \oplus \lfloor A \rfloor}\,\text{sm}\oplus \qquad \dfrac{\lfloor A \rfloor \xrightarrow{id} \lfloor A \rfloor}{\lfloor A \rfloor \otimes 1 \longrightarrow \lfloor A \rfloor}\,\text{sm}\otimes}{U \otimes 1 \longrightarrow V \oplus \lfloor A \rfloor}\,cut}{U \longrightarrow W}\,\text{sm}\otimes, \text{sm}\oplus$$

By Lemma 5.2 and symmetric monoidal coherence, this is equal to $\lfloor M \rfloor$.

Now for the soundness of CUT$\vee$. Because $\vee$L is expressed in terms of the constant $K_{\wedge \text{L}}$ (as explained in Section 4), it suffices to show that the interpretation of



(13)

is equal to $\lfloor L \rfloor$. If the domain and codomain of $\lfloor L \rfloor$ are U and W, respectively, then the interpretation of Net 13 is

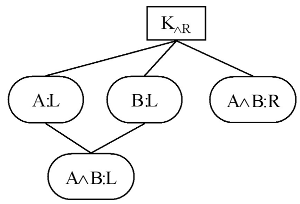
$$\dfrac{\dfrac{\dfrac{U \xrightarrow{\lfloor L \rfloor} W}{U \longrightarrow V \oplus (\lfloor A \rfloor \oplus \lfloor B \rfloor)}\,\text{sm}\oplus \qquad \dfrac{\lfloor A \rfloor \oplus \lfloor B \rfloor \xrightarrow{id} \lfloor A \rfloor \oplus \lfloor B \rfloor}{(\lfloor A \rfloor \oplus \lfloor B \rfloor) \otimes 1 \xrightarrow{id} \lfloor A \rfloor \oplus \lfloor B \rfloor}\,\text{sm}\otimes}{U \otimes 1 \longrightarrow V \oplus (\lfloor A \rfloor \oplus \lfloor B \rfloor)}\,cut}{U \longrightarrow W}\,\text{sm}\oplus, \text{sm}\otimes$$

By Lemma 5.2 and symmetric monoidal coherence, this is equal to $\lfloor L \rfloor$. Dually for CUT$\wedge$.

The soundness of CUT⊥ also follows from Lemma 5.2, by a straightforward argument, and dually for CUT⊤.

The soundness of W-MOVE (for $C \in \{\bot, \top\}$) follows immediately from symmetric monoidal coherence.

Proving the soundness of AX∧ boils down (by using CUTAX) to showing that the interpretation of



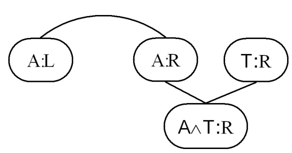
is $id_{\lfloor A \wedge B \rfloor}$. This follows directly from symmetric monoidal coherence. Dually for AX∨.

The soundness of TWIST⊤ and TWIST⊥ follows immediately from symmetric monoidal coherence. □

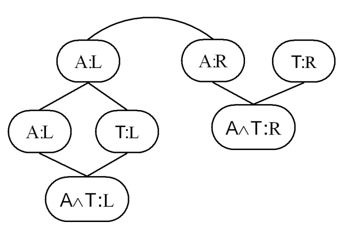
5.2. **Completeness of linear-net theories.**

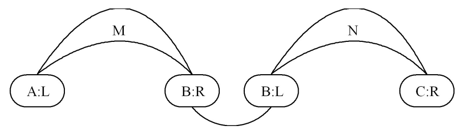
Theorem 5.4. *Every linear-net theory $\mathcal{T}$ forms a linearly distributive category $\mathcal{C}_{\mathcal{T}}$.*

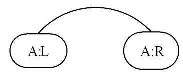
Proof. The objects of the linearly distributive category $\mathbf{C}_{\mathcal{T}}$ are the formulæ of $\mathcal{T}$. A morphism $A \longrightarrow B$ is a proof net with a door $A : L$, a door $B : R$, and no other doors. The categorical operators are defined in to Table 13. (The missing ones are given by duality and symmetry.) The associativity of composition is trivial and requires no equational law. The neutrality of the identity is stated by the law CUTAX. That the functor ⊗ preserves composition is stated by CUT∧, and that it preserves identities is stated by AX∧. Now for symmetric monoidal coherence. That $\sigma_{\otimes}$ is self-inverse follows from cutting $\sigma_{\otimes}^{A,B}$ against $\sigma_{\otimes}^{B,A}$, followed by an application of CUT∧ and a reverse application of AX∧. The same technique shows that $\alpha_{\otimes}$ is an isomorphism. The inverse of $\rho_{\otimes}$ is the net below. That $\rho_{\otimes} \circ \rho_{\otimes}^{-1} = id$ follows from cutting $\rho_{\otimes}$ and $\rho_{\otimes}^{-1}$ with cut formula $A \wedge \top$, followed by an application of CUT∧, then CUTAX, then CUT⊤.

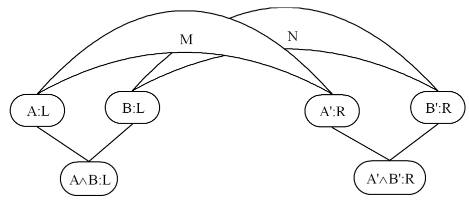


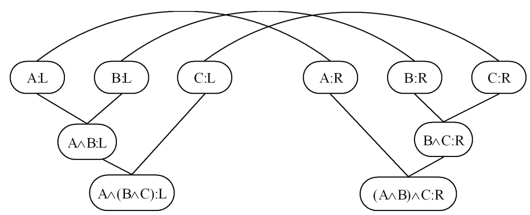
Proving that $\rho_{\otimes}^{-1} \circ \rho_{\otimes} = id$ is tricky; this confirms the old wisdom that units are often the most difficult aspects of the equational theory of sequent calculus (see *e.g.*, [2]). Cutting $\rho_{\otimes}^{-1}$ and $\rho_{\otimes}$ with cut formula A and applying CUTAX yields the net

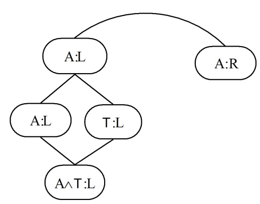


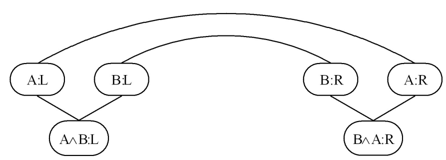
$(N : B \longrightarrow C) \circ (M : A \longrightarrow B) =$ 

$id_A =$ 

$(M : A \longrightarrow A') \otimes (N : B \longrightarrow B') =$ 

$\alpha_\otimes =$ 

$\rho_\otimes =$ 

$\sigma_\otimes =$ 

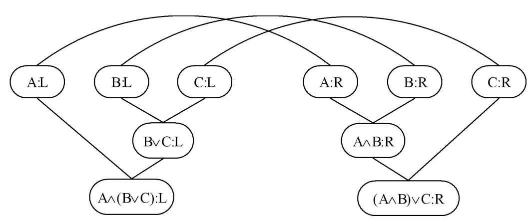
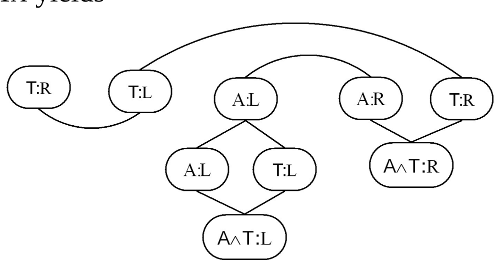
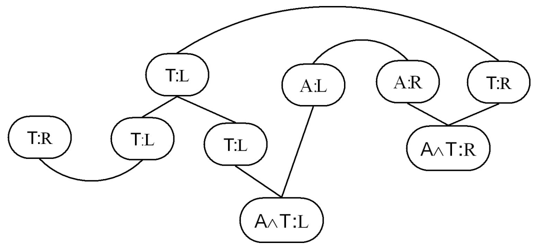
$\delta^L_L =$ 

TABLE 13. The linearly distributive category of linear nets

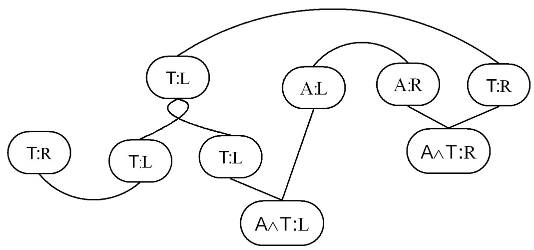
Another application of CUTAX yields



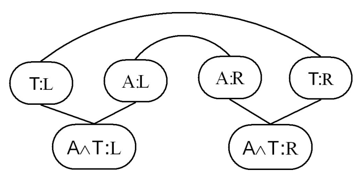
Applying W-MOVE yields



Now, literally and metaphorically, the twist: applying C-TWIST yields



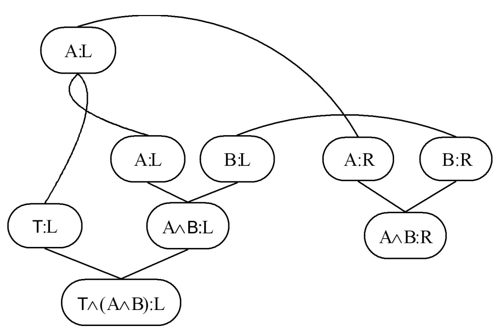
The point is that the left of the two occurrences of $\top : L$ is now the one which is "introduced by the weakening". Applying CUT$\top$ yields



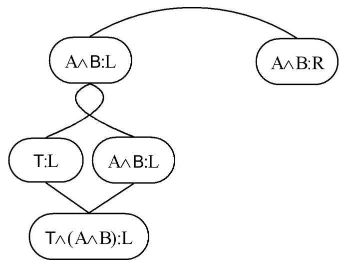
This net is the identity by AX$\wedge$.

The naturality of $\alpha_\otimes$ and $\sigma_\otimes$ is straightforward and relies only on CUT$\wedge$ and CUTAX. The naturality of $\rho_\otimes$ follows from CUT$\wedge$, CUTAX, and W-MOVE; we leave the details to the reader. Checking symmetric monoidal coherence (Diagrams 1–6) is also straightforward. So $\otimes$ forms a symmetric monoidal product. Dually for $\oplus$.

Now for the coherence laws involving the distributivity. To see that Diagram 7 commutes, note that $(\lambda_\otimes \oplus id) \circ \delta^L_L$, after applying CUT$\vee$, CUT$\wedge$, and CUTAX, is equivalent to



By re-wiring and AX$\wedge$, this is equivalent to



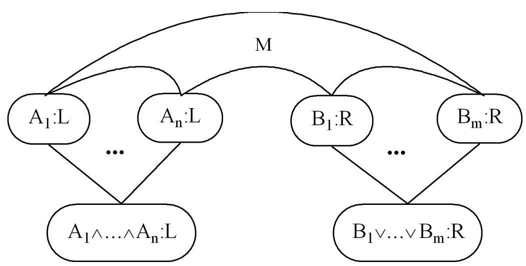
which is $\lambda_\otimes$ by definition.

Diagrams 7, 8, 9, and 10 follow from straightforward calculations using CUT$\vee$, CUT$\wedge$, and CUTAX. ☐

Now we turn towards completeness and initiality. Both results rely on the following lemma.

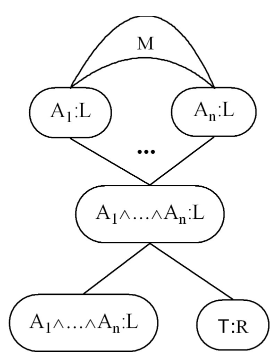
Lemma 5.5. *Let $\mathcal{T}$ be a linear-net theory, let M be a net of $\mathcal{T}$ with left doors $A_1, \ldots, A_n$ and right doors $B_1, \ldots, B_m$, and let $\mathbf{C}_\mathcal{T}\lfloor M \rfloor$ be the interpretation of M in $\mathbf{C}_\mathcal{T}$. If $n > 0$ and $m > 0$, then $\mathbf{C}_\mathcal{T}\lfloor M \rfloor$ is the equivalence class (w.r.t. equality $\equiv$ in $\mathcal{T}$) of*

(14)



where the left "rule" ending in $A_1 \wedge \ldots \wedge A_n$ stands for $n-1$ applications of the rule $\wedge$L, the right "rule" ending with $B_1 \vee \ldots \vee B_m$ stands for $m-1$ applications of $\vee$R. If $m = 0$ (and therefore $n > 0$), $\lfloor M \rfloor$ is the equivalence class of

(15)



and dually for the case $n = 0$.

Proof. By laborious induction over the size of M. ☐

Proposition 5.6. *In every linear-net theory $\mathcal{T}$, for any two nets M and N with matching sequences of doors, it holds that*

$$M \equiv N \text{ in } \mathcal{T} \quad \text{if and only if} \quad \mathbf{C}_\mathcal{T}\lfloor M \rfloor = \mathbf{C}_\mathcal{T}\lfloor N \rfloor$$

Proof. By Lemma 5.5, we have $\mathbf{C}_{\mathcal{T}} \lfloor M \rfloor = \mathbf{C}_{\mathcal{T}} \lfloor N \rfloor$ if and only if $M' \equiv N'$ in $\mathcal{T}$, where M' is the net in Picture 14 or 15 in Lemma 5.5, and similarly for N'. As can be easily checked, this holds if and only if $M \equiv N$ in $\mathcal{T}$. □

Now completeness follows immediately:

Theorem 5.7 (Completeness)**.** *Let $\mathcal{T}$ be a linear-net theory, and let M and N be nets of $\mathcal{T}$ with matching sequences of doors. If the equation $M \equiv N$ holds in every model of $\mathcal{T}$, then it is in $\mathcal{T}$.*

Theorem 5.8 (Initiality)**.** *For every model $\mathbf{C} \lfloor - \rfloor : \mathcal{T} \longrightarrow \mathbf{C}$ of a linear-net theory $\mathcal{T}$, there is a unique functor $F : \mathbf{C}_{\mathcal{T}} \longrightarrow \mathbf{C}$ that preserves all linearly distributive structure on the nose and makes the diagram below commute.*

$$\begin{array}{ccc} \mathbf{C}_{\mathcal{T}} & \xrightarrow{F} & \mathbf{C} \\ {\scriptstyle \mathbf{C}_{\mathcal{T}} \lfloor - \rfloor} \uparrow & \nearrow {\scriptstyle \mathbf{C} \lfloor - \rfloor} & \\ \mathcal{T} & & \end{array}$$

Proof. Because $\mathbf{C}_{\mathcal{T}}$ is bijective on objects, the object part of F is uniquely specified. Furthermore, every morphism of $\mathbf{C}_{\mathcal{T}}$ is in the image of $\mathbf{C}_{\mathcal{T}} \lfloor - \rfloor$: for if the morphism is the equivalence class of a net M (which by construction of $\mathbf{C}_{\mathcal{T}}$ has only on left door and one right door), then by Lemma 5.5 it is equal to $\mathbf{C}_{\mathcal{T}} \lfloor M \rfloor$. Because of this surjectivity of $\mathbf{C}_{\mathcal{T}} \lfloor - \rfloor$, F is also uniquely specified on morphisms. For F to be well-defined, we need that $M \equiv N$ in $\mathcal{T}$ implies $\mathbf{C}_{\mathcal{T}} \lfloor M \rfloor = \mathbf{C}_{\mathcal{T}} \lfloor N \rfloor$, but this is just the statement that $\mathbf{C} \lfloor - \rfloor$ is a model. It remains to show that F preserves all structure on the nose. This is a routine calculation in $\mathbf{C}$. □

5.3. **Adding negation.**

Definition 8. A *linear net with negation over a signature Σ with negation* is a net over Σ without occurrences CL and CR, such that WL occurs only with $B = \top$, and WR occurs only with $B = \bot$.

Definition 9. A *linear-net theory with negation $\mathcal{T}$* over a signature Σ with negation is a set of equalities $M \equiv N$ where M and N are linear nets with negation over Σ (with matching sequences of doors), such that $\equiv$ is a congruence which contains all instances of CUT$\wedge$, CUT$\vee$, CUT$\top$, CUT$\bot$, CUTAX, CUT$\neg$, W-MOVE for $C \in \{\bot, \top\}$, AX$\wedge$, AX$\vee$, AX$\neg$, TWIST$\top$, and TWIST$\bot$.

An *interpretation of a linear-net theory with negation* in a linearly distributive category with negation is defined like an interpretation in the absence of negation, plus the following two requirements: first, negation of formulæ is interpreted according to the rule

$$\lfloor \neg A \rfloor = \lfloor A \rfloor^*$$

Second, $K_{\neg \mathrm{L}}$ is interpreted by the map $\gamma^L : \lfloor A \rfloor^* \otimes \lfloor A \rfloor \longrightarrow 0$. Dually, $K_{\neg \mathrm{R}}$ is interpreted by the map $\tau^R : 1 \longrightarrow \lfloor A \rfloor \oplus \lfloor A \rfloor^*$. The following two lemmas are the key to soundness:

Lemma 5.9. *The equation below holds in every linearly distributive category with negation.*

$$\dfrac{\dfrac{1 \longrightarrow A \oplus A^* \qquad A^* \otimes A \longrightarrow 0}{1 \otimes A \longrightarrow A \oplus 0}\, cut}{A \longrightarrow A}\, \mathit{sm}\otimes, \mathit{sm}\oplus \qquad = \qquad id_A$$

Proof. After expressing the categorical operator cut in terms of the linear distributivity, the claim follows from Diagram 11 in the definition of a linearly distributive category with negation. □

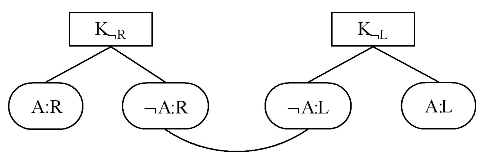
Lemma 5.10. *The equation below holds in every linearly distributive category with negation.*

$$\dfrac{1 \longrightarrow A \oplus A^* \cong A^* \oplus A \qquad A \otimes A^* \cong A^* \otimes A \longrightarrow 0}{\dfrac{1 \otimes A^* \longrightarrow A^* \oplus 0}{A^* \longrightarrow A^*}\ \textit{sm}\otimes,\ \textit{sm}\oplus}\ cut \qquad = \qquad id_{A^*}$$

Proof. After expressing the categorical operator cut in terms of the linear distributivity, the claim follows from Diagram 12 in the definition of a linear distributive category with negation. □

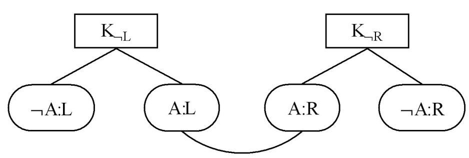
Proposition 5.11 (Soundness)**.** *Let $\lfloor - \rfloor$ be an interpretation of linear nets with negation over some signature Σ with negation. Then the judgments $M \equiv N$ such that $\lfloor M \rfloor = \lfloor N \rfloor$ form a linear-net theory with negation.*

Proof. Proving the soundness of CUT¬ boils down (by using CUTAX) to showing that the interpretation of



is $id_{\lfloor A \rfloor}$. This follows directly from Lemma 5.9.

Proving the soundness of AX¬ boils down (by using CUTAX) to showing that the interpretation of

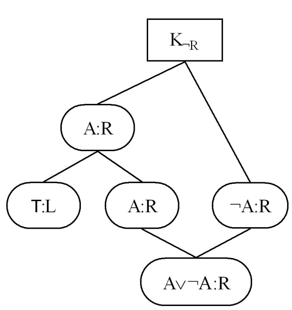


is $id_{\lfloor A^* \rfloor}$. This follows from Lemma 5.10. □

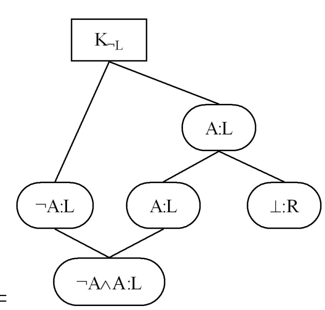
Theorem 5.12. *Every linear-net theory with negation forms a linearly distributive category with negation.*

Proof. We start with the category $\mathbf{C}_{\mathcal{T}}$ from Theorem 5.4 and define

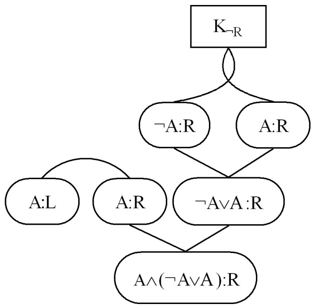
$\tau^R =$

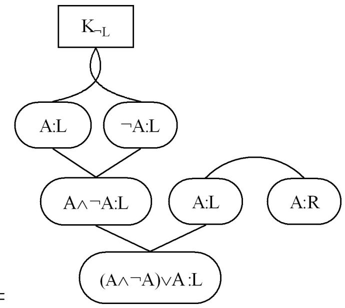


$\gamma^L =$

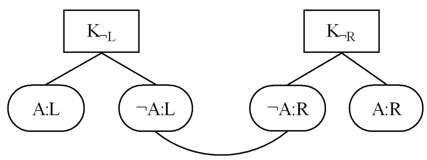


To check Diagram 11, we show that $\lambda_\oplus \circ (\gamma^R \oplus id) \circ \delta_L^L \circ (id \otimes \tau^L) \circ \rho_\otimes^{-1} = id$. To see this, note that

$(id \otimes \tau^L) \circ \rho_\otimes^{-1} =$ 

$\lambda_\oplus \circ (\gamma^L \oplus id) \quad =$ 

which follows from CUTAX, CUT$\wedge$, CUT$\top$ in case of the left equation, and from CUTAX, CUT$\vee$, CUT$\perp$ in case of the right equation. Connecting these two nets with δ_L^L and simplifying with CUT$\vee$, CUT$\wedge$, and CUTAX yields



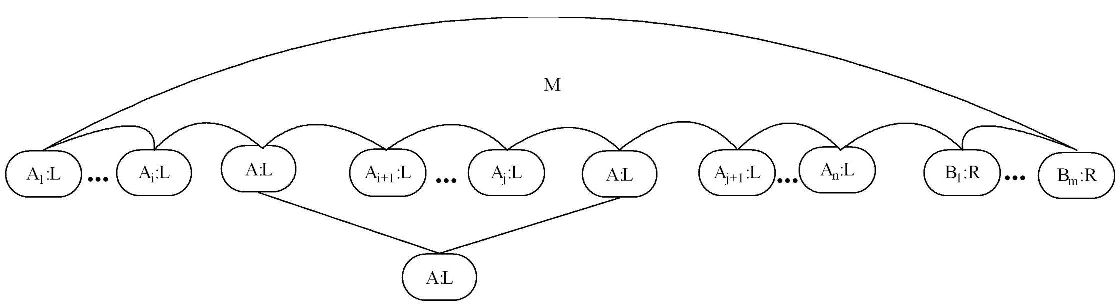
By CUT$\neg$ and CUTAX, this is equivalent to id_A.

Diagram 12 is checked in a similar way, except that (crucially!) the last step uses AX$\neg$ instead of CUT$\neg$. □

Lemma 5.5 carries over without problems to the situation with negation. Thus, completeness and initiality can be proved as in the negation-free case.

6. SEMANTICS OF WEAKENING AND CONTRACTION

The naïve semantics of weakening and contraction is evident: require the linearly distributive category with negation to have finite products and coproducts, and extend the notion of interpretation as follows: the net below, where we assume that the door $A : L$ is between A_k and A_{k+1},

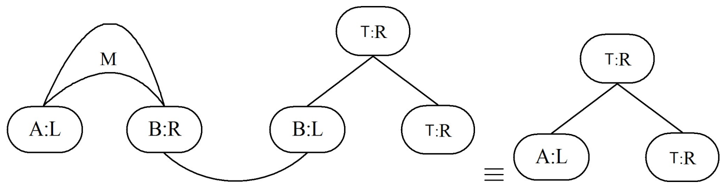


is interpreted by

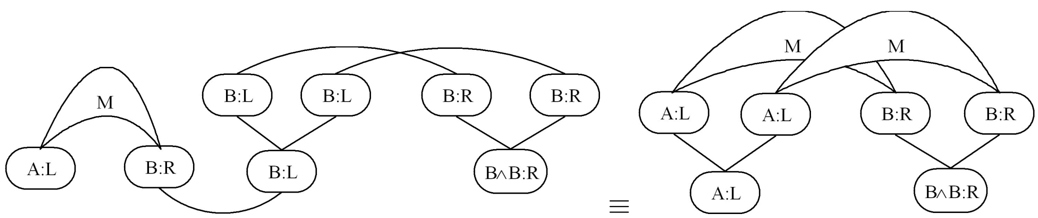
$$
\begin{aligned}
&\lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_k \rfloor \otimes \lfloor A \rfloor \otimes \lfloor A_{k+1} \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor \\
&\cong (\lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor) \otimes \lfloor A \rfloor \xrightarrow{id \otimes \Delta} (\lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor) \otimes (\lfloor A \rfloor \otimes \lfloor A \rfloor) \\
&\cong \lfloor A_1 \rfloor \otimes \cdots \lfloor A_i \rfloor \otimes \lfloor A \rfloor \otimes \lfloor A_{i+1} \rfloor \otimes \cdots \otimes \lfloor A_j \rfloor \otimes \lfloor A \rfloor \otimes \lfloor A_{j+1} \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor \\
&\xrightarrow{\lfloor M \rfloor} \lfloor B_1 \rfloor \oplus \cdots \oplus \lfloor B_m \rfloor
\end{aligned}
$$

where Δ is the diagonal associated with the finite products. The rule WL is interpreted similarly, except that the morphism which is pre-composed to $\lfloor M \rfloor$ is built by using the projection $\lfloor A \rfloor \longrightarrow 1$ instead of Δ. Dually for CR and WR.

But net-theories are not complete for models with finite products and coproducts: for example, the terminal object would necessitate the law

(16) 

which in categorical form is $\langle\rangle \circ f = \langle\rangle$. Owing to CUTW, all net-theories have the left-to right reduction $\preccurlyeq$. But to escape the collapse cause by Lafont's example, we had to drop the converse $\succcurlyeq$. Similarly, finite products would necessitate the law

(17) 

which in categorical form is essentially $\Delta \circ f = (f \otimes f) \circ \Delta$. Because of CUTC, all net-theories have the left-to right reduction $\preccurlyeq$. But the converse $\succcurlyeq$ does not generally hold. (As we shall see, $\mathbf{Rel}$ is a counter-model.)

Therefore, we shall weaken the requirements imposed on diagonals, projections, co-diagonals, and co-projections. We shall proceed as follows:

Section 6.1: To each object A, we add a monoid structure with multiplication $\nabla : A \oplus A \longrightarrow A$ and unit $[] : 0 \longrightarrow A$. Dually, we add a co-monoid structure with co-multiplication $\Delta : A \longrightarrow A \otimes A$ and co-unit $A \longrightarrow 1$. This structure is enough to interpret weakening and contraction;

Section 6.2: However, we shall see that this structure does not generally admit CUTW and CUTC. To address this issue, we shall introduce an order-enrichment, together with some conditions about the interaction between the monoids, the co-monoids, and the order. We call the resulting structures *classical categories*. Finally, we shall prove soundness, completeness, and initiality of classical categories with respect to the classical sequent calculus.

In particular, we shall see that:

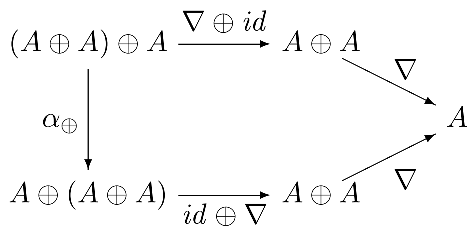
(1) There is a remarkably close correspondence between the coherence laws for nets in Tables 9 and 12 and the equational laws for the monoids and co-monoids. This will become evident in the proof of Theorem 6.2;
(2) The proof of soundness (Theorem 6.4) is unusually informative: it reveals that both CUTW and CUTC combine two very different categorical manipulations in one step.

6.1. **Monoids and co-monoids.** In this section, we shall define what it means for a linearly distributive category to *have monoids and co-monoids*, and show that the linearly distributive category built from a net theory has such structure.

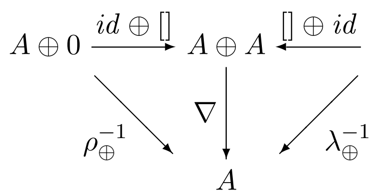
First we recall what it means for a symmetric monoidal category to *have monoids*[1].

Definition 10. Let $\mathbf{C} = (\mathbf{C}, \oplus, 0)$ be a symmetric monoidal category. A *symmetric monoid* in $\mathbf{C}$ is given by an object A, together with two morphisms $\nabla_A : A \oplus A \longrightarrow A$ and $[]_A : 0 \longrightarrow A$, satisfying the usual equations

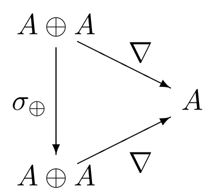
(18)



(19)

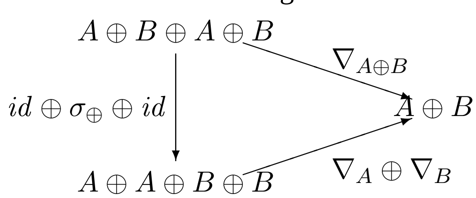


(20)

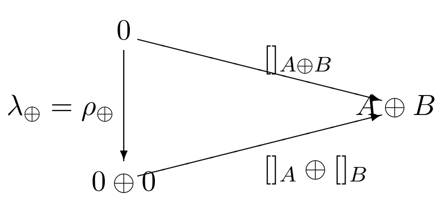


$\mathbf{C}$ *has monoids* if there is a chosen symmetric monoid $(A, \nabla_A, []_A)$ for every object A, compatible with the symmetric monoidal structure in the following sense:

(21)



(22)



(23) $$[]_0 = id_0 : 0 \longrightarrow 0$$

(In the last diagram, some obvious associativity isomorphisms have been omitted.)

Note that Equations 21 and 22 simply state that the monoidal operations at compound carriers $A \oplus B$ are defined *pointwise* in terms of the operations of the carriers A and B. Equation 23 can be

[1]This definition is taken from from [22], except that Selinger's paper deals with the more general case of *premonoidal* categories and uses the terminology "has co-diagonals" instead of "has monoids".

seen as the nullary case of Equation 22. The nullary case of Equation 21, $\nabla_0 = \lambda_\oplus$, is easily seen to be derivable.

Note also that we do not require the families of maps ∇_A and $[]_A$ to be natural in A. In other words, we do not require all maps of $\mathbf{C}$ to be monoid-homomorphisms. Instead, we call a morphism $f : A \longrightarrow B$ *copyable* if the diagram

$$\begin{array}{ccc} A \oplus A & \longrightarrow & A \\ {\scriptstyle f\oplus f}\downarrow & & \downarrow{\scriptstyle f} \\ B \oplus B & \longrightarrow & B \end{array}$$

commutes, and *discardable* if the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ {\scriptstyle []}\downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & B \end{array}$$

commutes. This terminology was introduced by [23] and also used in [22, 8, 13]. (However, those publications deal with the semantics of functional programming languages and natural-deduction calculi, and the *premonoidal* categories they use differ significantly from linearly distributive categories.) The map $f : A \longrightarrow B$ is called *focal* if it is both copyable and discardable.

Note that to say that f is copyable is to say that f preserves the monoidal multiplication ∇—that is, f is a homomorphism of semigroups. To say that f is discardable is to say that f preserves the monoidal unit $[]$. To say that f is focal is to say that f is a homomorphism of monoids.

The *focus* of $\mathbf{C}$ is defined to be the lluf[2] subcategory of focal maps.

Dually, we define what it means for a symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, 1)$ to *have comonoids*, and the notions *co-copyable*, *co-discardable*, *co-focal*, and *co-focus*. (Caution: the notions "copyable", "discardable", and "focal", in [23, 8, 13] correspond to our notions "co-copyable", "co-discardable", and "co-focal". However, our terminology agrees with [22].)

As observed in [22], the focus of a symmetric monoidal category with monoids is closed under $\oplus$ and contains all structural maps ($\alpha_\oplus$, $\lambda_\oplus$, $\rho_\oplus$, $\sigma_\oplus$, ∇, and $[]$), as well as the co-projections

$$\iota_1 : A \xrightarrow{\rho_\oplus^{-1}} A \oplus 0 \xrightarrow{id \oplus []} A \oplus B$$
$$\iota_2 : A \xrightarrow{\lambda_\oplus^{-1}} 0 \oplus A \xrightarrow{[] \oplus id} A \oplus B$$

Furthermore, the focus has a canonical finite coproduct structure:

Lemma 6.1. *In the focus, the object 0 is initial, and $\oplus$ is a coproduct with injections ι_1 and ι_2 and co-pairing*

$$[f, g] = A \oplus B \xrightarrow{f \oplus g} C \oplus C \xrightarrow{\nabla} C$$

In fact, the focus is the largest subcategory on which $\oplus$ restricts to a coproduct.

Example 1. The category $\mathbf{Rel}$, whose objects are sets, and whose morphisms $A \longrightarrow B$ are subsets of the set-theoretic product $A \times B$. The functors $\otimes$ and $\oplus$ coincide: the sets $A \otimes B$ and $A \oplus B$ are simply $A \times B$. The units 1 and 0 are the singleton set $\{*\}$. Given $f : A \longrightarrow B$ and $f' : A' \longrightarrow B'$, the morphism $f \otimes f' = f \oplus f' : A \otimes A' \longrightarrow B \otimes B'$ is defined to be

$$\{((x, x'), (y, y')) : (x, y) \in f \wedge (x', y') \in f'\}$$

[2] A subcategory of $\mathbf{C}$ is called *lluf* if it has all objects of $\mathbf{C}$.

The set A^* is simply A. The law $1 \longrightarrow A \oplus A^*$ of the excluded middle is $\{(*, (x, x)) : x \in A\}$, and dually for the contradiction law $A^* \otimes A \longrightarrow 0$. The monoidal and co-monoidal operations are

$$\nabla_A = \{((x, x), x) : x \in A\} \qquad []_A = \{(*, x) : x \in A\}$$
$$\Delta_A = \{(x, (x, x)) : x \in A\} \qquad \langle\rangle_A = \{(x, *) : x \in A\}$$

Example 2. Every Boolean lattice $\mathbf{B}$. The objects are the elements of $\mathbf{B}$, and a morphism $A \longrightarrow B$ is a pair (A, B) such that $A \leq B$. Composition and identities are trivial. The functor $\otimes$ is given by the infimum operator $\wedge$, and $\oplus$ by the supremum operator $\vee$. The object 1 is the greatest element $\top$, and 0 is the smallest element $\bot$. The linear distributivity is the law $A \wedge (B \vee C) \leq (A \wedge B) \vee C$ which holds in every Boolean lattice. The operator $(-)^*$ is the complement operator of $\mathbf{B}$. The monoidal multiplication ∇_A is the idempotency law $A \vee A = A$, and the monoidal unit is the inequality $\bot \leq A$. Dually for the co-monoidal structure.

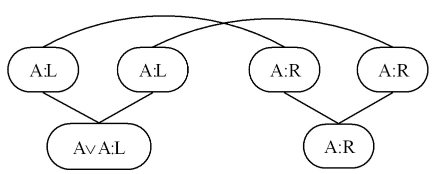
Obviously, given two linearly distributive categories $\mathbf{C}$ and $\mathbf{C}'$ with negation, monoids, and co-monoids, the product category $\mathbf{C} \times \mathbf{C}'$ has again such structure. In particular, we have

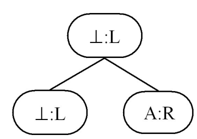
Example 3. $\mathbf{Rel} \times \mathbf{B}$ for every Boolean lattice $\mathbf{B}$. In such a category, we have $\oplus \neq \otimes$ (other than in $\mathbf{Rel}$), and there are hom-spaces with more that one element (other than in $\mathbf{B}$).

Theorem 6.2. *For every net theory $\mathcal{T}$, the linearly distributive category $\mathcal{C}_{\mathcal{T}}$ has monoids and co-monoids.*

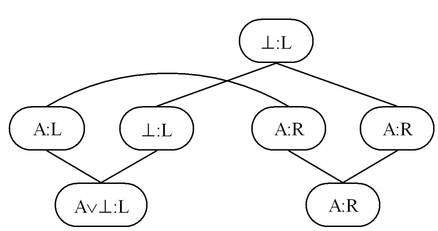
The proof of this theorem highlights the close correspondence between the equational laws for the monoids and co-monoids and the coherence laws for nets in Tables 9 and 12.

Proof of Theorem 6.2. We start with the linearly distributive category from Theorem 5.4. We prove that $\mathbf{C}$ has monoids (the existence of co-monoids follows by duality). Define

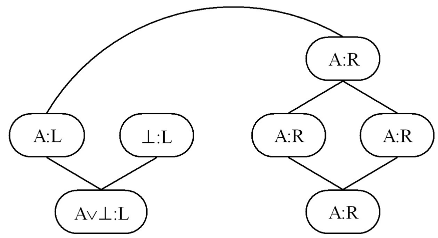
$\nabla :=$ 

$[] :=$ 

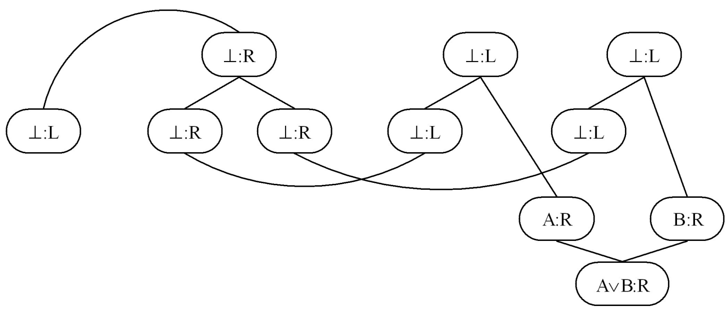
To see that Diagram 18 commutes, note that both nets, after simplification with CUT∨ and CUTAX, are the same up to C-ASSOC. Diagram 20 commutes because both nets, after CUT∨ and CUTAX, are the same up to C-TWIST. Diagram 21 commutes because both nets, after CUT∨ and CUTAX, are the same up to C∨. To see that Diagram 19 commutes, note that the net $\nabla \circ ([] \oplus id)$, after CUT∨ and CUTAX, is



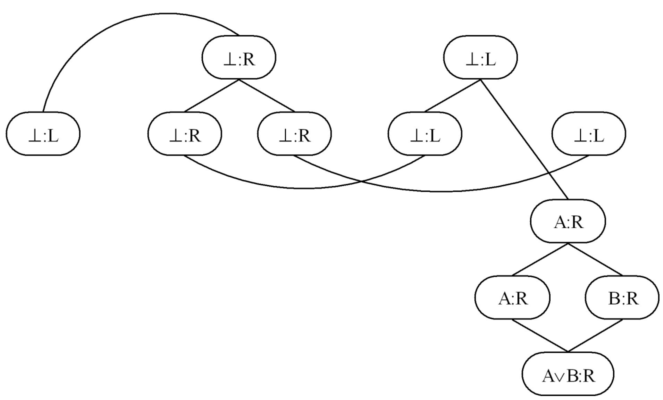
Re-wiring yields



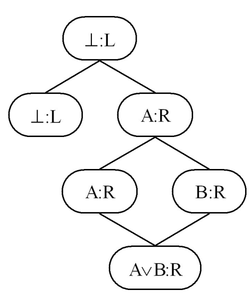
which is equal to $\rho_{\oplus}^{-1}$ by WC. To see that Diagram 22 commutes, note that the net $([]_A \oplus []_B) \circ \lambda_{\oplus}$, after CUT∨, is



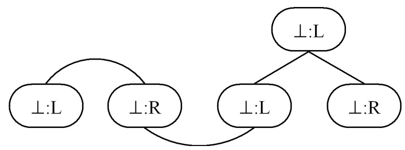
Re-wiring yields



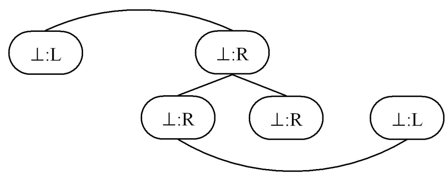
After CUT⊥ and CUTAX, this is



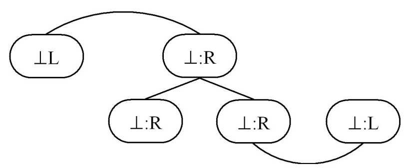
which is equal to $[]_{A\oplus B}$ by W$\vee$. To see that $[]_0 = id_0$, note that $[]_0$ is equal to



by CUTAX. By re-wiring, this is equivalent to



But the $\bot : R$ to which the cut is connected is not the one introduced by the weakening! However, by Rule TWIST$\bot$, we get



By Rule CUT$\top$, this is equivalent to id_1. □

6.2. **The order enrichment.** Linearly distributive categories with negation, monoids, and co-monoids provide interpretations of net theories. However, they are not good enough to count as models because they may fail to admit CUTC or CUTW. To see how CUTC may fail to be a semantic equality, note that in any net theory $\mathcal{T}$ we have (where Δ_B denotes the evident net corresponding to the co-multiplication $B \longrightarrow B \otimes B$, and, recalling that 0 in $\mathbf{Rel}$ is the singleton set, $[]_B$ denotes the evident net corresponding to the unit $0 \longrightarrow B$)

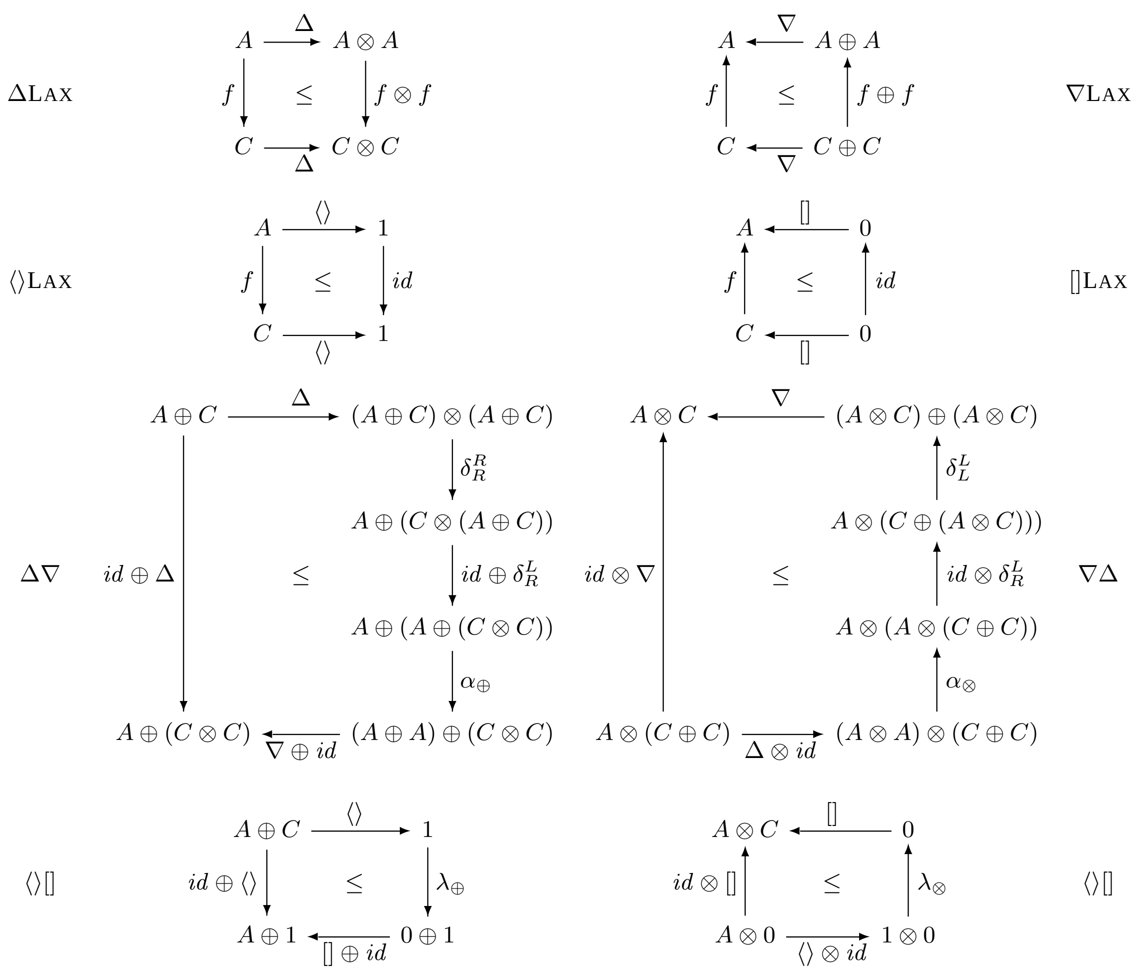
$$\Delta_B \circ []_B \preccurlyeq^{\text{CUTC}} ([]_B \otimes []_B) \circ \Delta_\bot$$

modulo the equivalences CUT$\wedge$, CUT$\vee$, and CUTAX. (This is the left-to-right reduction in Equation 17 with $M = []_B$.) But, in $\mathbf{Rel}$, the interpretation of the redex turns out to be $\{(*, (x, x)) : x \in B\}$, whereas the interpretation of the reduct turns out to be $\{(*, (x, y)) : x, y \in B\}$.

To see how CUTW may fail to be a semantic equality, suppose that every interpretation admits CUTW. Then the reductions in Lafont's example would be interpreted by equalities, and therefore any two nets with only door $A : R$ would have the same interpretation. But in $\mathbf{Rel}$, as explained above, we have $\Delta_B \circ []_B \neq ([]_B \otimes []_B) \circ \Delta_\bot$, for non-trivial B, and thus we have two different morphisms $0 \longrightarrow B \otimes B$, both of which are denotable by nets.

To model CUTC and CUTW adequately, we introduce an order-enrichment. By *ordered category*, we mean a category together with a partial order on every hom-space, such that the composition of morphisms is monotonic. (In the jargon of enriched category theory, a "$\mathbf{po}$-enriched category", where $\mathbf{po}$ stands for the category of partial orders and monotonic functions.)

Definition 11. A *classical category* is an ordered linearly distributive category $\mathbf{C}$ with negation, monoids, and co-monoids, such that $\otimes$, $\oplus$, and the negation functor are monotonic in all arguments, and the following inequalities hold (where f ranges over arbitrary morphisms of $\mathbf{C}$):



Example 4. $\mathbf{Rel}$, where the order between morphism is the set-theoretic inclusion of morphisms. Also, every Boolean lattice, where the order between morphisms is trivial (because hom-spaces have at most one element).

Obviously, given two classical categories $\mathbf{C}$ and $\mathbf{C}'$ the product category $\mathbf{C} \times \mathbf{C}'$, which we already observed to be a linearly distributive category with monoids, co-monoids and negation, is again the classical category with

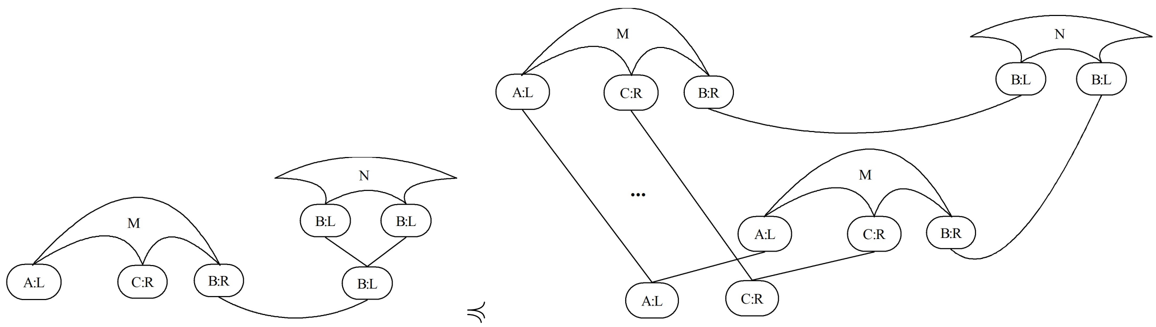
$$(f, f') \leq (g, g') \iff f \leq g \text{ and } f' \leq g'$$

In particular, we have

Example 5. $\mathbf{Rel} \times \mathbf{B}$ is a classical category for every Boolean lattice $\mathbf{B}$.

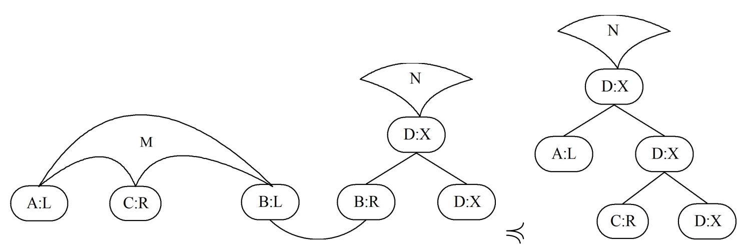
A more substantial model, based on the Geometry of Interaction, is presented in [7]. (The details are beyond the scope of the present paper.)

The use of the eight inequality laws will be explained precisely in the soundness and completeness proofs. However, we shall first explain these laws in a more intuitive way. The law ΔLAX, which states that Δ is a "lax natural transformation", is essentially the left-to-right reduction in Equation 17. (As observed earlier, the converse $\not\succcurlyeq$ does not generally hold.) This reduction is possible owing to CUTC. But CUTC is more powerful, in a subtle way: note that the net M in Equation 17 has only one right door, B, which is the cut formula. However, the rule CUTC allows M to have further right doors which are not used in the cut, for example



The key point is that the right door C is copied, and we must undo this with a *right* contraction. This compensation has nothing to do with the law ΔLAX. It is captured by the law $\Delta\nabla$, which states that "copying too much and then compensating by co-copying may increase the denotation".

Similarly, the law $\langle\rangle$LAX, which states that $\langle\rangle$ is a lax natural transformation, is essentially the left-to-right reduction in Equation 16. This reduction is possible owing to CUTW, but CUTW is more powerful. Consider the following instance of CUTC:

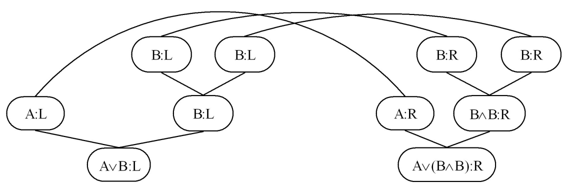


Here the key point is that the right door C is discarded, and we must compensate for this with a *right* weakening. This compensation has nothing to do with the law $\langle\rangle$LAX. It is captured by the law $\langle\rangle[]$, which states that "discarding too much and then compensating by co-discarding may increase the denotation".

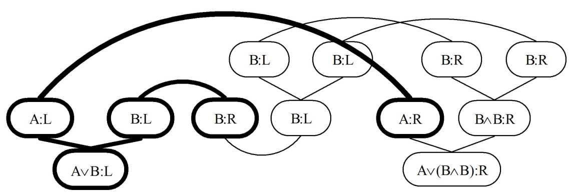
Theorem 6.3. *Every net theory forms a classical category.*

Proof. Let $\mathcal{T}$ be a net theory. By Theorem 6.2, we know that $\mathbf{C}_{\mathcal{T}}$ is a linearly distributive category with negation, monoids, and co-monoids. Diagram ΔLAX is, modulo CUT$\wedge$ and CUTAX, and instance of CUTC. Dually for Diagram ∇LAX. Diagram $\langle\rangle$LAX is, modulo CUTAX and $\langle\rangle_1 \equiv id_1$, an instance of CUTW. Dually for Diagram $[]$LAX.

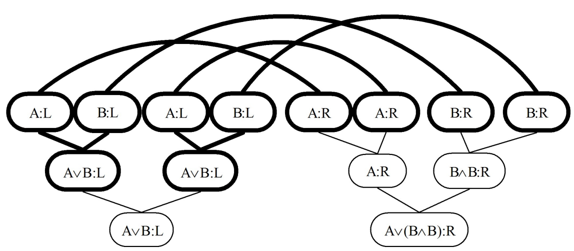
Now for Diagram $\Delta\nabla$. We have



$id \oplus \Delta =$



$\equiv$ CUTAX

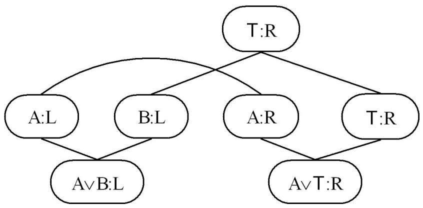


$\preccurlyeq$ CUTC (the bold subnet got copied)

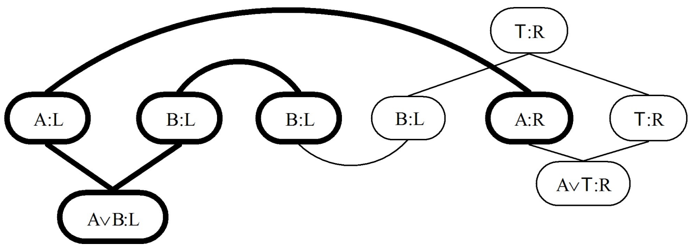
$\equiv (\nabla \oplus id) \circ \alpha_{\oplus} \circ (id \oplus \delta) \circ \delta \circ \Delta$ CUT$\wedge$, CUT$\vee$, CUTAX, C-TWIST

The key point is that CUTC introduces the "compensating contraction" for the right A. Dually for Diagram $\nabla\Delta$.

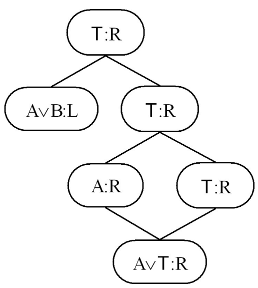
Finally, we show Diagram $\langle\rangle[]$. We have



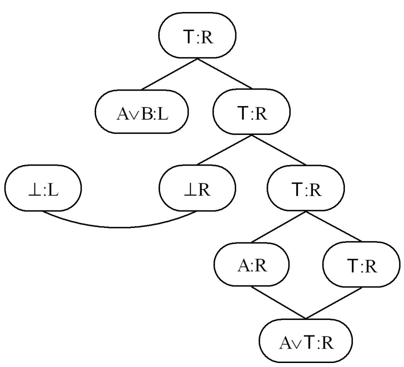
$id \oplus \langle\rangle =$



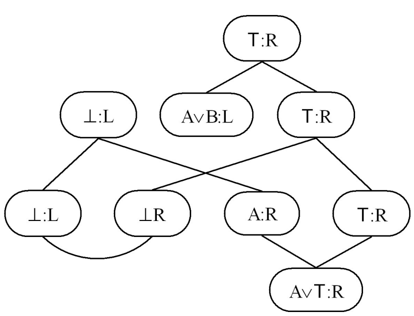
$\equiv$ CUTAX



$\preccurlyeq$ CUTW (the bold subnet got discarded)



$\equiv$ CUT$\bot$



$\equiv$ re-wiring

$\equiv ([] \oplus id) \circ \lambda_{\oplus} \circ \langle\rangle$ CUT$\vee$, CUTAX

The key point is that CUTW destroys the axiom link between the left A and the right A and introduces "compensating weakening" instead. Dually for Diagram $[]\langle\rangle$. □

Now we finally get to the first main theorem:

Theorem 6.4 (Ordered soundness)**.** *For every classical-category interpretation $\lfloor - \rfloor$ of nets over a signature Σ, the judgments $M \preccurlyeq N$ such that $\lfloor M \rfloor \leq \lfloor N \rfloor$ form a net theory.*

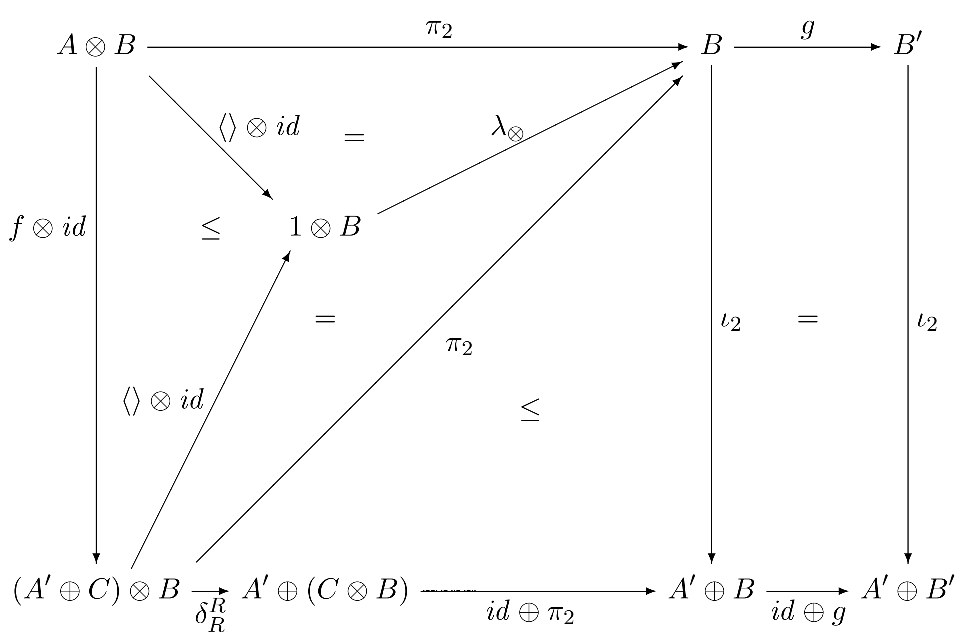
The proof of this theorem obviates the necessity of all eight inequalities in the definition of a classical category.

Proof. Because we already have soundness for linear-net theories with negation (Proposition 5.11), and because we have the monotonicity of $\otimes$, $\oplus$, and negation, which implies that $\preccurlyeq$ is compatible (in the sense of Definition 5), it remains to prove the soundness of the inequalities CUTW and CUTC, and the equalities C-ASSOC, C-CROSS, C-TWIST, WC, W-MOVE, W$\wedge$, W$\vee$, C$\wedge$, and C$\vee$. For each of the equalities, the two sides correspond to different ways of pre-composing projections and diagonals, or different ways of post-composing co-projections and co-diagonals. But it is evident that, for each equality, these two ways are semantically the same, because of the finite products on the focus and the finite coproducts on the focus (Lemma 6.1).

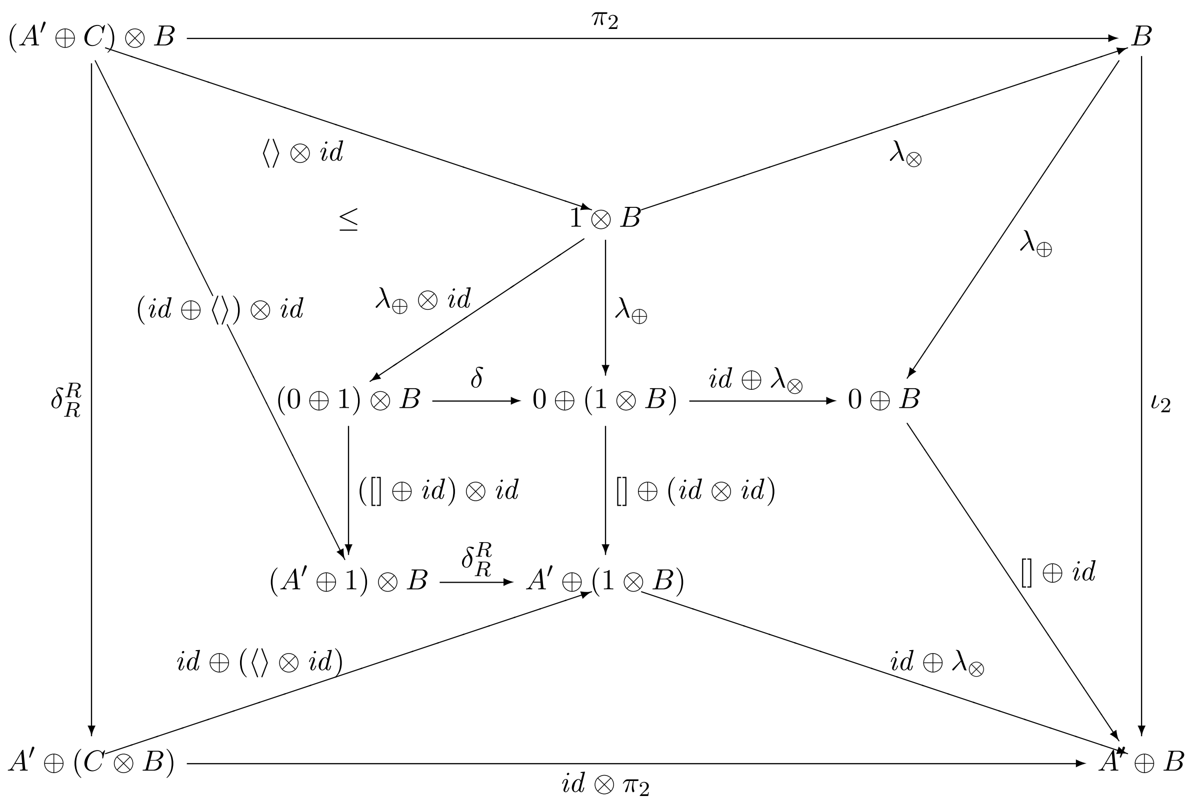
Now for the soundness of CUTW. Without loss of generality, let $Z = L$ in the presentation of CUTW in Table 8. It is easy to see that the soundness follows from the law

$$\frac{A \xrightarrow{f} A' \oplus C \qquad C \otimes B \xrightarrow{\pi_2} B \xrightarrow{g} B'}{A \otimes B \longrightarrow A' \oplus B'}\, cut \leq A \otimes B \xrightarrow{\pi_2} B \xrightarrow{g} B' \xrightarrow{\iota_2} A' \oplus B' \tag{24}$$

To see the that Inequality 24 holds, consider the following diagram:



The lower-left leg is the left side of Inequality 24 (by definition of the categorical operator cut). The inequality in the leftmost triangle holds because of Condition $\langle\rangle$LAX in the definition of a classical category. The two equalities in the triangles hold by definition of π_2. The equality in the square holds by naturality of ι_2. So it remains to prove the inequality in the rightmost triangle. To see this, consider the diagram below.



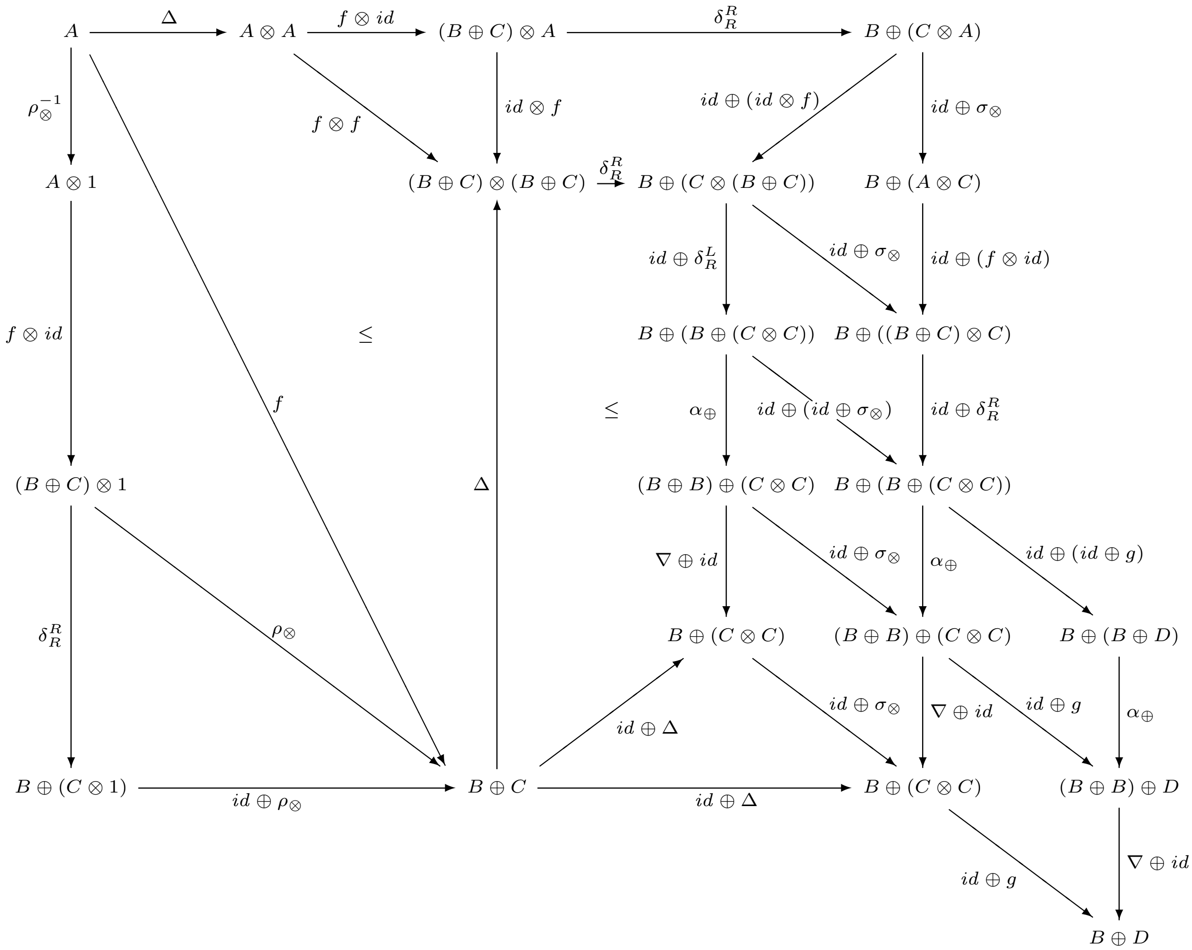
The square containing the inequality follows from Condition $\langle\rangle[]$ in the definition of a classical category. All other parts of the diagram commute: the top and bottom triangle by definition of π_2. The rightmost triangle by definition of ι_2. The leftmost square and the innermost square commute owing to the naturality of δ^R_R. The innermost triangle is, up to symmetry, the coherence law 7 in the definition of a linearly distributive category. The upper-right square holds owing to the naturality of $\lambda_\oplus$, and the lower-right square because $\oplus$ is functorial.

Now for the soundness of CUTC. Without loss of generality, let $X = L$ in the presentation of CUTW in Table 8. It is easy to see that the soundness follows from the law

(25)
$$\frac{A \xrightarrow{f} B \oplus C \qquad C \otimes 1 \xrightarrow{\rho_\otimes} C \xrightarrow{\Delta} C \otimes C \xrightarrow{g} D}{\dfrac{A \otimes 1 \longrightarrow B \oplus D}{A \longrightarrow B \oplus D}\,\rho_\otimes^{-1}}\,cut$$

$$\leq \quad \frac{A \xrightarrow{f} B \oplus C \qquad \dfrac{\dfrac{A \xrightarrow{f} B \oplus C \qquad C \otimes C \xrightarrow{g} D}{A \otimes C \longrightarrow B \oplus D}}{C \otimes A \longrightarrow B \oplus D}\,\sigma_\otimes}{\dfrac{A \otimes A \longrightarrow B \oplus (B \oplus D)}{A \longrightarrow B \oplus D}\,\Delta, \alpha_\oplus, \nabla}\,cut$$

To see the that Inequality 25 holds, consider the following diagram:



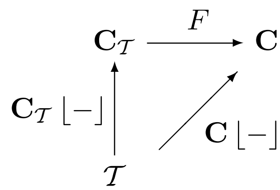
The lower-left leg is the left side of Inequality 25, and the upper-right leg is the right side. The left inequality is Condition ΔLAX from the definition of a classical category. The right inequality is Condition $\Delta\nabla$. The other parts of the diagram commute for straightforward reasons. □

Now for the second main theorem:

Theorem 6.5 (Ordered completeness)**.** *Let $\mathcal{T}$ be a net theory, and let M and N be nets of $\mathcal{T}$ with matching sequences of doors. If we have $\lfloor M \rfloor \preccurlyeq \lfloor N \rfloor$ for every interpretation of $\mathcal{T}$ in a classical category, then the judgment $M \preccurlyeq N$ is in $\mathcal{T}$.*

Proof. Lemma 5.5 can be extended to the case with weakening and contraction. Thus, we get an extended version of Proposition 5.6, for the case where $\mathcal{T}$ is a net theory and $\mathcal{C}_{\mathcal{T}}$ is the classical category from Theorem 6.3. Now the claim follows immediately. □

Theorem 6.6 (Initiality)**.** *For every classical-category model* $\mathbf{C}\,\lfloor - \rfloor : \mathcal{T} \longrightarrow \mathbf{C}$ *of a net theory $\mathcal{T}$, there is a unique functor* $F : \mathbf{C}_{\mathcal{T}} \longrightarrow \mathbf{C}$ *that preserves all classical-category structure on the nose* *and makes the diagram below commute.*



Proof. As already mentioned in the ordered-completeness proof, Lemma 5.5, and consequently Proposition 5.6, can be extended to the case with weakening and contraction. So the proof of initiality works as in the linear case, except that it remains to prove that the functor F, which is already known to preserve the linearly distributive structure and negation, also preserves the monoids, co-monoids, and the order. This follows from straightforward calculations in the classical category $\mathbf{C}$. $\square$

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