A PATTERN-MATCHING CALCULUS FOR $*$-AUTONOMOUS CATEGORIES

ABSTRACT. This article sums up the details of a linear λ-calculus that can be used as an internal language of $*$-autonomous categories. The coherent isomorphisms associated with the autonomous (= symmetric monoidal) structure are represented by pattern matching, and the inverse to the evident natural transformation $\eta_A : A \longrightarrow ((A \multimap \top) \multimap \top)$ is represented by a type-indexed family of constant $\mathcal{C}_A : ((A \multimap \top) \multimap \top) \multimap A$.

The calculus presented in this article is no great novelty. It is essentially DILL [1] minus additives, plus extended pattern-matching and a type-indexed family of constants $\mathcal{C}_A : ((A \multimap \top) \multimap \top) \multimap A$. It can also be seen as Hasegawa's DCLL [2] minus the additive functions space $\rightarrow$, plus pattern matching and the multiplicative conjunction $\otimes$. The main purpose of our calculus is the rapid verification of equalities in autonomous and $*$-autonomous categories. To this end, we emphasize pattern matching, which in is very useful in practice, more strongly than in [1] or [2]. (A note to the impatient: the translation of categorical expressions into our calculus is presented in Table 5.)

Autonomous category is another word for "symmetric monoidal category". *$*$-autonomous category* is an autonomous category together with an object $\perp$, such that the evident natural transformation $\eta_A : A \longrightarrow (A \multimap \perp) \multimap \perp$ has an inverse.

We shall present a calculus for autonomous categories, the *autonomous calculus*, and discuss the equational laws for the family of constants $\mathcal{C}_A : ((A \multimap \top) \multimap \top) \longrightarrow A$ which represents the inverse of the natural transformation $\eta : A \longrightarrow ((A \multimap \top) \multimap \top)$.

The syntax of the autonomous calculus is as follows:

Types	$A, B ::= A \multimap B \mid A \otimes B \mid \top \mid b$
Patterns	$P, Q ::= x \mid (P, Q) \mid ()$
Terms	$M, N ::= x \mid \lambda P : A.M \mid MN \mid (M, N) \mid () \mid c^A$

where b ranges over base types and c^A over constants of type A. The typing rules for patterns are

$$\frac{}{x : b \vdash_p x : b} \qquad \frac{}{x : A \multimap B \vdash_p x : A \multimap B}$$

$$\frac{\Gamma \vdash_p P : A \quad \Delta \vdash_p Q : B}{\Gamma, \Delta \vdash_p (P, Q) : A \otimes B} \text{ if } \mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Delta) = \emptyset \qquad \frac{}{\vdash_p () : \top}$$

(The capital greeks letters Γ, Δ range over non-repetitive sequences $x_1 : A_1, x_2 : A_1, \ldots, x_n : A_n$ of typed variables, and $\mathrm{dom}(\Gamma)$ is the set of variables that occur in Γ.) Note that in a pattern we allow only variables whose type is *not* of the form $\top$ or $A \otimes B$. Also, note that for every type A, there a unique pattern P_A for which some judgement $\Gamma \vdash_p P_A : A$ is derivable—up to renaming the variables in $\mathrm{dom}(\Gamma)$.

The typing rules for terms are presented in Table 1. The notation "$\Gamma \sharp \Delta$" stands for any merging of

$$\frac{}{x : A \vdash x : A}$$

$$\frac{\Gamma, \Delta \vdash M : B}{\Gamma \vdash \lambda P : A.M : A \multimap B}\ \Delta \vdash_p P : A$$

$$\frac{\Gamma \vdash M : A \multimap B \qquad \Delta \vdash N : A}{\Gamma \sharp \Delta \vdash MN : B}\ \text{if } \mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Delta) = \emptyset$$

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma \sharp \Delta \vdash (M, N) : A \otimes B}\ \text{if } \mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Delta) = \emptyset$$

$$\frac{}{\vdash () : \top}$$

TABLE 1. Typing judgments of the autonomous calculus

Γ and Δ. For example, we allow

$$\frac{x_1 : A_1, x_2 : A_2 \vdash M : A \qquad y_1 : B_2, y_2 : B_2 \vdash N : B}{y_1 : B_1, x_1 : A_1, y_2 : B_2, x_2 : A_2 \vdash (M, N) : A \otimes B}$$

Thus, the permutation rule

$$\frac{\Gamma \vdash M : A}{\Gamma' \vdash M : A}\ \text{if } \Gamma' \text{ is a permutation of } \Gamma$$

is derivable by induction over typing judgements. The autonomous calculus is linear in the sense that, in a derivable judgement $\Gamma \vdash M : A$, every variable in Γ has exactly one free occurrence in M. So every judgement $\Gamma \vdash M : A$ has a unique derivation, because in the binary typing rules (application and pairing) it is determined for each free variable whether it comes from the left premise, or from the right.

We use $\mathit{let}\ P : A\ \mathit{be}\ M\ \mathit{in}\ N$ as "syntactic sugar" for $(\lambda P : A.N)M$. The equations of the autonomous calculus are presented in Table 2.

A *signature* Σ consists of a collection of base types b and a collection of constants c_A.

Definition 1. An *autonomous theory* over a signature Σ is a set of judgments $\Gamma \vdash M \equiv N : A$, where $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$ are derivable typing judgements over Σ, such that $\equiv$ is a congruence that contains all equations described in Table 2.

Proposition 0.1. *The equations in Table 2 are interderivable with the equations in Table 3.*

Proposition 0.2. *In every autonomous theory, the β-rule $\mathit{let}\ x\ \mathit{be}\ M\ \mathit{in}\ N \equiv N[M/x]$ holds.*

Proof. By induction over the derivation of N. $\square$

ASSOC	$let\ Q\ be\ (let\ P\ be\ L\ in\ M)\ in\ N \equiv let\ P\ be\ L\ in\ let\ Q\ be\ M\ in\ N$	if $\mathrm{FV}(P) \cap \mathrm{FV}(N) = \emptyset$
ID	$let\ P\ be\ M\ in\ P \equiv M$	
$\mathrm{PAT}_{\otimes}$	$let\ (P_1, P_2)\ be\ (M_1, M_2)\ in\ N \equiv let\ P_1\ be\ M_1\ in\ let\ P_2\ be\ M_2\ in\ N$	if $\mathrm{FV}(P_1) \cap \mathrm{FV}(M_2) = \emptyset$
$\mathrm{PAT}'_{\otimes}$	$let\ (P_1, P_2)\ be\ (M_1, M_2)\ in\ N \equiv let\ P_2\ be\ M_2\ in\ let\ P_1\ be\ M_1\ in\ N$	if $\mathrm{FV}(P_2) \cap \mathrm{FV}(M_1) = \emptyset$
$\mathrm{PAT}_{\top}$	$let\ ()\ be\ ()\ in\ N \equiv N$	
η	$\lambda P.MP \equiv M$	if $\mathrm{FV}(P) \cap \mathrm{FV}(M) = \emptyset$
LET_{λ}	$let\ Q\ be\ M\ in\ \lambda P.N \equiv \lambda P.let\ Q\ be\ M\ in\ N$	if $\mathrm{FV}(P) \cap \mathrm{FV}(M) = \emptyset$

TABLE 2. Equations of the autonomous calculus

ID	$let\ P\ be\ M\ in\ P \equiv M$	
$\mathrm{PAT}_{\otimes}$	$let\ (P_1, P_2)\ be\ (M_1, M_2)\ in\ N \equiv let\ P_1\ be\ M_1\ in\ let\ P_2\ be\ M_2\ in\ N$	if $\mathrm{FV}(P_1) \cap \mathrm{FV}(M_2) = \emptyset$
$\mathrm{PAT}_{\top}$	$let\ ()\ be\ ()\ in\ N \equiv N$	
η	$\lambda P.MP \equiv M$	if $\mathrm{FV}(P) \cap \mathrm{FV}(M) = \emptyset$
LET_{λ}	$let\ Q\ be\ M\ in\ \lambda P.N \equiv \lambda P.let\ Q\ be\ M\ in\ N$	if $\mathrm{FV}(P) \cap \mathrm{FV}(M) = \emptyset$
$\mathrm{LET}_{\mathrm{app}}$	$let\ P\ be\ L\ in\ MN \equiv M(let\ P\ be\ L\ in\ N)$	if $\mathrm{FV}(P) \cap \mathrm{FV}(M) = \emptyset$
$\mathrm{LET}'_{\mathrm{app}}$	$let\ P\ be\ L\ in\ MN \equiv (let\ P\ be\ L\ in\ M)N$	if $\mathrm{FV}(P) \cap \mathrm{FV}(N) = \emptyset$
$\mathrm{LET}_{\mathrm{pair}}$	$let\ P\ be\ L\ in\ (M, N) \equiv (M, let\ P\ be\ L\ in\ N)$	if $\mathrm{FV}(P) \cap \mathrm{FV}(M) = \emptyset$
$\mathrm{LET}'_{\mathrm{pair}}$	$let\ P\ be\ L\ in\ (M, N) \equiv (let\ P\ be\ L\ in\ M, N)$	if $\mathrm{FV}(P) \cap \mathrm{FV}(N) = \emptyset$

TABLE 3. Alternative equations for the autonomous calculus

Now for the semantics of the autonomous calculus.

Definition 2. An *interpretation* of typing judgments over a signature Σ in an autonomous category $\mathbf{C}$ sends

- each type A of $\mathcal{T}$ homomorphically to an object $\lfloor A \rfloor$ of $\mathbf{C}$,
- each environment $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ to the object $\lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor$ (where $\otimes$ associates to the left, say, and the empty product is $\top$), and
- each judgement $\Gamma \vdash M : A$ to a morphism $\lfloor \Gamma \vdash M : A \rfloor : \lfloor \Gamma \rfloor \longrightarrow \lfloor A \rfloor$ according to the rules in Table 4.

(Thus, an interpretation is uniquely determined by its effect on base types and constants.) A *model* of an autonomous theory $\mathcal{T}$ is an interpretation $\lfloor - \rfloor$ such that, for every equation $\Gamma \vdash M \equiv N : A$ in $\mathcal{T}$, it holds that $\lfloor \Gamma \vdash M : A \rfloor = \lfloor \Gamma \vdash N : A \rfloor$.

$$\lfloor x : A \vdash x : A \rfloor \quad = id_{\lfloor A \rfloor}$$

$$\frac{\lfloor \Gamma, \Delta \vdash M : B \rfloor \quad = f : \lfloor \Gamma, \Delta \rfloor \longrightarrow \lfloor B \rfloor}{\lfloor \Gamma \vdash \lambda P : A.M : A \multimap B \rfloor \quad = \lambda \left(\lfloor \Gamma \rfloor \otimes \lfloor A \rfloor \cong \lfloor \Gamma, \Delta \rfloor \xrightarrow{f} \lfloor B \rfloor \right)}$$

$$\frac{\begin{array}{ll}\lfloor \Gamma \vdash M : A \multimap B \rfloor & = f : \lfloor \Gamma \rfloor \longrightarrow \lfloor A \rfloor \multimap \lfloor B \rfloor \\ \lfloor \Delta \vdash N : A \rfloor & = g : \lfloor \Delta \rfloor \longrightarrow \lfloor A \rfloor\end{array}}{\lfloor \Gamma \sharp \Delta \vdash MN : B \rfloor \quad = \lfloor \Gamma, \Delta \rfloor \cong \lfloor \Gamma \rfloor \otimes \lfloor \Delta \rfloor \xrightarrow{f \otimes g} (\lfloor A \rfloor \multimap \lfloor B \rfloor) \otimes \lfloor A \rfloor \xrightarrow{apply} \lfloor B \rfloor}$$

$$\frac{\begin{array}{ll}\lfloor \Gamma \vdash M : A \rfloor & = f : \lfloor \Gamma \rfloor \longrightarrow \lfloor A \rfloor \\ \lfloor \Delta \vdash N : B \rfloor & = g : \lfloor \Delta \rfloor \longrightarrow \lfloor B \rfloor\end{array}}{\lfloor \Gamma \sharp \Delta \vdash (M, N) : A \otimes B \rfloor \quad = \lfloor \Gamma, \Delta \rfloor \cong \lfloor \Gamma \rfloor \otimes \lfloor \Delta \rfloor \xrightarrow{f \otimes g} \lfloor A \rfloor \otimes \lfloor B \rfloor}$$

$$\lfloor \vdash () : \top \rfloor \quad = id_{\top}$$

TABLE 4. Semantics of the autonomous calculus

Proposition 0.3 (Soundness). *For every interpretation $\lceil - \rceil$ of typing judgements over a signature Σ, the well-typed equations $\Gamma \vdash M \equiv N : A$ such that $\lfloor \Gamma \vdash M : A \rfloor = \lfloor \Gamma \vdash N : A \rfloor$ form an autonomous theory.*

Theorem 0.4 (Completeness). *Suppose that $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$ are typing judgments of a theory $\mathcal{T}$. If the equation $\Gamma \vdash M \equiv N : A$ holds in every model of $\mathcal{T}$, then it is in $\mathcal{T}$.*

Proof. We present a term model of $\mathcal{T}$. A *categorical language* is given by the grammar

Types	$A, B ::= A \multimap B \mid A \otimes B \mid \top \mid b$
Terms	$f, g ::= id_A \mid g \circ f \mid g \otimes f \mid \alpha_{A,B,C} \mid \lambda_A \mid \rho_A \mid \sigma_{A,B} \mid \lambda f \mid apply_{A,B} \mid h^{A \longrightarrow B}$

(with the evident typing rules) where b ranges over base types and h ranges over generators of type $A \longrightarrow B$. A *reverse interpretation* sends

- every type A of the categorical language to a type $\lceil A \rceil$ of the autonomous calculus, homomorphically (*i.e.*, it does nothing but exchange base types)
- every term $f : A \longrightarrow B$ of the categorical language to a term $\lceil f \rceil : \lceil A \rceil \multimap \lceil B \rceil$ of the autonomous calculus, according to Table 5.

It is easy to check that, for every reverse interpretation, the categorical language, modulo the induced equality $f \equiv g \iff \lceil f \rceil \equiv \lceil g \rceil$, forms an autonomous category.

To prove completeness, we start with the categorical language whose base types are those of $\mathcal{T}$, and whose generators are of the form $h_c : \top \longrightarrow A$ for every constant $c : A$ of $\mathcal{T}$. Let $\lceil - \rceil$ be the reverse interpretation that sends each base type to itself, and each generator $h_c : \top \longrightarrow A$ to

$$\begin{array}{ll}
\lceil id_A : A \longrightarrow A \rceil & = \lambda P : \lceil A \rceil .P \\
\lceil g \circ f : A \longrightarrow C \rceil & = \lambda P : \lceil A \rceil . \lceil g \rceil (\lceil f \rceil P) \\
\lceil f_1 \otimes f_2 : A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2 \rceil & = \lambda (P_1, P_2) : \lceil A_1 \rceil \otimes \lceil A_2 \rceil .(\lceil f_1 \rceil P_1, \lceil f_2 \rceil P_2) \\
\lceil \alpha : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C \rceil & = \lambda (P, (Q, R)) : \lceil A \rceil \otimes (\lceil B \rceil \otimes \lceil C \rceil).((P, Q), R) \\
\lceil \lambda : \top \otimes A \longrightarrow A \rceil & = \lambda ((), P) : \top \otimes \lceil A \rceil .P \\
\lceil \rho : A \otimes \top \longrightarrow A \rceil & = \lambda (P, ()) : \lceil A \rceil \otimes \top .P \\
\lceil \sigma : A \otimes B \longrightarrow B \otimes A \rceil & = \lambda (P, Q) : \lceil A \rceil \otimes \lceil B \rceil .(Q, P) \\
\lceil \lambda f : A \longrightarrow (B \multimap C) \rceil & = \lambda P : \lceil A \rceil .\lambda Q : \lceil B \rceil . \lceil f \rceil (P, Q) \\
\lceil apply_{A,B} : (A \multimap B) \otimes A \multimap B \rceil & = \lambda (f, P) : (\lceil A \rceil \multimap \lceil B \rceil) \otimes A. f P
\end{array}$$

TABLE 5. "Reverse interpretation" (used e.g. in the term-model construction)

$\lambda() : \top.c$. Then it is not hard to show that the induced autonomous category forms an initial model of $\mathcal{T}$. □

Now we make the step from autonomous categories to $*$-autonomous categories. A $*$-autonomous category is an autonomous category together with an object $\perp$ such that, for every object A, the evident map $\eta_A : A \longrightarrow ((A \multimap \perp) \multimap \perp)$ has an inverse. In the autonomous calculus, η_A is $\lambda P : A.\lambda k : A \multimap \perp.kP$. We represent its inverse by a constant $\mathcal{C}_A : ((A \multimap \perp) \multimap \perp) \multimap A$. Thus, the categorical equations

$$\mathcal{C}_A \circ \eta_A = id_A$$
$$\eta_A \circ \mathcal{C}_A = id_{(A \multimap \perp) \multimap \perp}$$

(after simplification) correspond to

$$\lambda P : A.\mathcal{C}_A(\lambda k : A \multimap \perp.kP) \equiv \lambda P : A.P$$
$$\lambda h : (A \multimap \perp) \multimap \perp.\lambda k : A \multimap \perp.k(\mathcal{C}_A h) \equiv \lambda h : (A \multimap \perp) \multimap \perp.h$$

in the autonomous calculus. The first equation is interderivable with

$$\mathcal{C}_A(\lambda k : A \multimap \perp.kM) \equiv M \qquad \mathcal{C}\text{-}\textsc{App}$$

where M ranges over terms of type A (with no free occurrence of k), and the second equation is interderivable with

$$K(\mathcal{C}_A M) \equiv MK \qquad \mathcal{C}\text{-}\textsc{Eta}$$

where M ranges over terms of type $(A \multimap \perp) \multimap \perp$ and K ranges over terms of type $A \multimap \perp$. (Of course, we must have $\mathrm{FV}(M) \cap \mathrm{FV}(K) = \emptyset$.) Thus we obtain:

Theorem 0.5. *The autonomous calculus together with a type $\perp$ and a type-indexed family of constants $\mathcal{C}_A : ((A \multimap \perp) \multimap \perp) \multimap A$ satisfying the rules $\mathcal{C}$-*App *and $\mathcal{C}$-*Eta *is sound and complete for $*$-autonomous categories.*

The use of the law $\mathcal{C}$-ETA is unpleasantly restricted because K must have type $A \multimap \perp$ rather than $A \multimap B$ for arbitrary B. This restriction is lifted by the following law:

$$N(\mathcal{C}_A M) \equiv \mathcal{C}_B(\lambda k : B \multimap \perp . M(k \circ N)) \qquad \mathcal{C}\text{-NAT}$$

To see that this law holds, consider

$$\begin{aligned} N(\mathcal{C}_A M) &\equiv \mathcal{C}_B(\lambda k . B \multimap \perp . k(N(\mathcal{C}M))) && \mathcal{C}\text{-APP} \\ &= \mathcal{C}_B(\lambda k . B \multimap \perp . (k \circ N)(\mathcal{C}M)) \\ &\equiv \mathcal{C}_B(\lambda k . B \multimap \perp . M(k \circ N)) && \mathcal{C}\text{-ETA} \end{aligned}$$

(The law $\mathcal{C}$-NAT is called so because it essentially encodes the fact that the type-indexed family $\mathcal{C}_A$ represents a natural transformation.)

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