Intuitionistic logic

Motivation for intuitionistic logic

- As hinted earlier, proof by contradiction (RAA) is contentious.
- As shown before, (RAA) is interderivable with the law of the excluded middle

$$\frac{}{A \vee \neg A}\, LEM\,.$$

- We shall now see an example why LEM (and therefore RAA) is contentious.

Motivation for intuitionistic logic

Proposition. There exist two irrational numbers a, b such that a^b is rational.

Constructivism

- The proof we have seen is deemed to be not **constructive**.
- An attack on the law of the excluded middle was launched by the famous mathematician-logician L.E.J. Brouwer in the early 1900's.
- Brouwer's mathematics and logics are called **intuitionistic**.

In this context, the traditional non-constructive mathematics and logics are called **classical**.

Heyting interpretation

- The idea in constructive logic is that we can only consider a statement to be true if we have a proof for it.
- This idea is made precise by Heyting's interpretation of proofs:

Heyting interpretation

- A proof of $A \wedge B$ is a pair (Φ, Ψ) where Φ is a proof of A and Ψ is a proof of B.
- A proof of $A \vee B$ is a proof of A or a proof of B.
- A proof of $A \rightarrow B$ is a method for turning a proof of A into a proof of B.
- A proof of $\forall x.A$ is a method for turning any witness t, into a proof of $A[t/x]$.
- A proof of $\exists x.A$ consists of a witness t and a proof Φ of $A[t/x]$.

ND and Heyting interpretation

The gist of the Heyting interpretation is captured by the natural deduction rules **minus RAA**:

ND and Heyting interpretation: $\wedge$

$$\dfrac{\begin{array}{cc}\Phi & \Psi \\ \vdots & \vdots \\ A & B\end{array}}{A \wedge B}\wedge i$$

Given a proof Φ of A and a proof Ψ of B, we have a proof of $A \wedge B$.

$$\dfrac{\begin{array}{c}\Phi \\ \vdots \\ A \wedge B\end{array}}{A}\wedge e \qquad \dfrac{\begin{array}{c}\Phi \\ \vdots \\ A \wedge B\end{array}}{B}\wedge e$$

Given a proof Φ of $A \wedge B$, we have a proof of A and a proof of B.

So, to have a proof of $A \wedge B$ is to have a proof of A and a proof of B.

ND and Heyting interpretation: $\rightarrow$

$$\cfrac{\begin{array}{c}[A]\\ \vdots\ \Phi \\ B\end{array}}{A \rightarrow B} \rightarrow i$$

Given a method Φ for turning a proof of A into a proof of B, we have a proof of $A \rightarrow B$.

$$\cfrac{\begin{array}{c}\Phi\\ \vdots \\ A \rightarrow B\end{array} \qquad \begin{array}{c}\Psi\\ \vdots \\ A\end{array}}{B} \rightarrow e$$

Given a proof Φ of $A \rightarrow B$, we have a method for turning any proof Ψ of A into a proof of B.

So, to have a proof of $A \rightarrow B$ is to have a method for turning any proof of A into a proof of B.

ND and Heyting interpretation: $\forall$

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ A\end{array}}{\forall x.A}\,\forall i$$

Given a proof of A for an arbitrary x (i.e., a method for proving $A[t/x]$ for any t), we have a proof of $\forall x.A$.

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \forall x.A\end{array}}{A[t/x]}\,\forall e$$

Given a proof of $\forall x.A$, we have a method for proving of $A[t/x]$ for any t.

(Warning: the side conditions are omitted in the above presentation of the rules.) So, to have a proof of $\forall x.A$ is to have a method for proving $A[t/x]$ for any t.

ND and Heyting interpretation: $\vee$

$$\dfrac{\begin{matrix}\Phi\\ \vdots\\ A\end{matrix}}{A \vee B}\,\vee i \qquad \dfrac{\begin{matrix}\Phi\\ \vdots\\ B\end{matrix}}{A \vee B}\,\vee i$$

Given a proof of Φ of A (or of B), we have a proof of $A \vee B$.

$$\dfrac{\begin{matrix}\Phi\\ \vdots\\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\,\Psi_1\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\,\Psi_2\\ C\end{matrix}}{C}\,\vee e$$

Given a proof Φ of $A \vee B$ and methods Ψ_1 resp. Ψ_2 for turning proofs of A resp. B into proofs of C, we have a proof of C.

The disjunction property

- Introduction and elimination rules for $\vee$ do **not** imply the **disjunction property**, which states that

  if $\vdash A \vee B$, then $\vdash A$ or $\vdash B$.

- To see this, note that in classical propositional logic, we have neither $\vdash p$ nor $\vdash \neg p$ for an atomic formula p.
- But the disjunction property holds for intuitionistic logic, as we shall see later.

ND and Heyting interpretation: $\exists$

$$\dfrac{\begin{matrix}\Phi \\ \vdots \\ A[t/x]\end{matrix}}{\exists x.A}\ \exists i$$

Given a proof Φ of $A[t/x]$ for some witness t, we have a proof of $\exists x.A$.

$$\dfrac{\exists x.A \qquad \begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{B}\ \exists e$$

Given a proof Φ of $A[t/x]$, and a method for turning a proof of A (for arbitrary x) into a proof of B, we get a proof of B.

(Warning: the side conditions are omitted in the above presentation of the rules.)

The existence property

- Introduction and elimination rules for $\exists$ do **not** imply the **existence property**, which states that

$$\text{if } \vdash \exists x.A, \text{ then } \vdash A[t/x] \text{ for some } t.$$

- But the existence property holds for intuitionistic logic.

Ex falso quodlibet

The elimination rule for $\bot$ is contentious, but not as contentious as RAA. (As seen earlier, RAA implies $\bot e$; the converse is false, as we shall see.)

$$\frac{\begin{array}{c}\Phi \\ \vdots \\ \bot\end{array}}{A}\ \bot e$$

If Φ is a proof of a contradiction , we are allowed to turn this into a proof of any formula A.

This rule is allowed in intuitionistic logic, but not in minimal logic.

Summary of ND for IL

For simplicity, we shall focus on propositional logic.

$$\frac{A \quad B}{A \wedge B}\wedge i \qquad \frac{A \wedge B}{A}\wedge e \qquad \frac{A \wedge B}{B}\wedge e$$

$$\frac{A}{A \vee B}\vee i \qquad \frac{B}{A \vee B}\vee i \qquad \frac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\vee e \qquad \frac{\bot}{A}\bot e$$

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\rightarrow i \qquad \frac{A \rightarrow B \quad A}{B}\rightarrow e$$

Alternative version

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge i \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge e \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge e$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee i \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee i$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee e \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \bot e$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow e$$

Semantics of IL?

- $\Gamma \vdash A$ is provable in ND for **classical** propositional logic iff $\Gamma \models A$ in the sense of the truth-table semantics.
- The absence of RAA from IL suggests that IL proves fewer judgments $\Gamma \vdash A$, and is therefore incomplete w.r.t. the truth-table semantics.
- Is there a semantics w.r.t. which IL **is** complete?

Kripke models of IL

Remarkably, a variation of Kripke models for modal logic also works for IL. Three changes are enough:

1. The accessibility relation R is a **preorder**, i.e. reflexive and transitive. We shall write $\leq$ instead of R.
2. The labelling function is required to be **monotonic**, i.e. $L(x) \subseteq L(y)$ whenever $x \leq y$.
3. We shall need to change the forcing semantics of implication.

Heuristic motivation

- An idealized mathematician (traditionally called the “creative subject”) explores the possible worlds.
- The preorder can be seen to describe (branching) time: $x < y$ means that world y is later than world x.
- The mathematician can only move forward in time; along the way, she discovers true facts.
- If she knows a fact to be true at world x, she also knows it to be true in any later world. (That explains why the labelling function is monotonic.)

Kripke models for IL

Definition. A **(Kripke) model** of propositional IL consists of

1. a set W, whose elements are called **worlds**;
2. a preorder $\leq$ on W;
3. a monotonic labelling function $L : W \to P(Atoms)$.

Semantics of $\wedge$, $\vee$, $\bot$

The semantics of $\wedge$, $\vee$, $\bot$, and of atomic formulæ, is the same as in basic modal logic:

$$x \Vdash A \wedge B \quad \text{iff} \quad x \Vdash A \text{ and } x \Vdash B$$

$$x \Vdash A \vee B \quad \text{iff} \quad x \Vdash A \text{ or } x \Vdash B$$

$$x \nVdash \bot$$

$$x \Vdash p \quad \text{iff} \quad p \in L(x)$$

Semantics of $\rightarrow$

- One can know $A \rightarrow B$ to be true without knowing whether A or B are true.
- However, it does not suffice to look only at the present world: one must know that no later discovery can make $A \rightarrow B$ false.

This motivates the following semantics of $\rightarrow$:

$x \Vdash A \rightarrow B$ iff for all y with $x \leq y$, if $y \Vdash A$ then $y \Vdash B$.

Semantics of $\rightarrow$: discussion

Let x be a world, and let p and q be atomic formulæ.

1. If q is true at x, then $x \Vdash p \rightarrow q$.
2. If p is true and q is false at x, then $x \nVdash p \rightarrow q$.
3. If both p and q are false at x, we must look into the future.

Semantics of $\neg$

As before, we define

$$\neg A = (A \to \bot).$$

Thus

$$x \Vdash \neg A \quad \text{iff} \quad \text{for all } y \text{ with } x \leq y \text{ we have } y \nVdash A.$$

That is, we know $\neg A$ if A never becomes true.

Double negation

Lemma. In every Kripke model for IL, it holds for every world x that

$$x \Vdash \neg\neg A$$

if and only if

for all $y \geq x$ there is a $z \geq y$ such that $z \Vdash A$.

Proof. See lecture.

Some non-valid formulæ

The following formulæ, which are valid in classical logic, are not valid in IL:

1. $\neg\neg p \rightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$
4. $(p \rightarrow q) \rightarrow (\neg p \vee q)$.