Sequent calculus vs. natural deduction

Sequent calculus and ND

Theorem. A sequent $\Gamma \vdash A$ is derivable in the sequent calculus if and only if it is derivable in natural deduction.

Sequent calculus and ND

Let's write

$$\Gamma \vdash_{seq} \Delta$$

if some sequent $\Gamma \vdash \Delta$ is derivable in the sequent calculus, and

$$\Gamma \vdash_{ND} A$$

if some sequent $\Gamma \vdash A$ is derivable in ND. So the theorem states

$$\Gamma \vdash_{seq} A \quad \text{iff} \quad \Gamma \vdash_{ND} A.$$

From ND to sequent calculus

- We show

$$\Gamma \vdash_{seq} A \quad \Leftarrow \quad \Gamma \vdash_{ND} A$$

  by induction on the size of the proof of $\Gamma \vdash_{ND} A$.
- We proceed by case split on the last rule used in the proof of $\Gamma \vdash_{ND} A$.

Axioms

- Case (1): the ND proof is

$$\frac{}{\Gamma, A \vdash_{ND} A}\ Ax.$$

The sequent proof is

$$\frac{\dfrac{}{A \vdash_{seq} A}\ Ax}{\Gamma, A \vdash_{seq} A}\ LW.$$

ND introduction rules

- Case (2): the last rule of the ND proof is an introduction rule:

  $$\rightarrow i, \wedge i, \vee i.$$

  These cases are essentially handled by the right introduction rules

  $$R \rightarrow, R\wedge, R\vee.$$

  of the sequent calculus.

Elimination rules

- Case (3): the last rule of the ND proof is an elimination rule.

$$\wedge e, \rightarrow e, \vee e, \perp e.$$

They are handled by **left introduction rules plus Cut** (see lecture).

Reductio ad absurdum

- Case (4): the last rule of the ND proof is

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A} \; RAA.$$

See lecture.

From sequent calculus to ND

- We still have to show

$$\Gamma \vdash_{seq} A \quad \Rightarrow \quad \Gamma \vdash_{ND} A. \qquad (1)$$

- One shows by (a tedious) induction on the sequent proof that

$$\Gamma \vdash_{seq} A_1, \ldots, A_m \quad \Rightarrow \quad \Gamma, \neg A_1, \ldots, \neg A_m \vdash_{ND} \bot$$

Then (**??**) follows from the case $m = 1$ by RAA.

Soundness and completeness

Theorem. The sequent $\Gamma \vdash \Delta$ is provable in the sequent calculus if and only if $\Gamma \models \Delta$.

Proof. The claim follows from soundness & completeness for ND: suppose that $\Delta = A_1, \ldots, A_m$. Then

$$\begin{aligned} \Gamma \vdash_{seq} \Delta &\iff \Gamma, \neg A_1, \ldots, \neg A_m \vdash_{ND} \bot \\ &\iff \Gamma, \neg A_1, \ldots, \neg A_m \models \bot \\ &\iff \Gamma \models A_1, \ldots, A_m. \end{aligned}$$

The subformula property

Definition. An inference rule

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

has the **subformula property** if every formula in the Γ_i or Δ_j is a subformula of Γ or Δ.

- The subformula property is nice, because it limits the possible hypotheses of $\Gamma \vdash \Delta$.
- So it helps **proof search**.

The cut rule

$$\frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3} \; Cut$$

- Needed for translating ND proofs into sequent proofs.

Gentzen's famous **Hauptsatz** (main theorem):

Theorem. Every sequent-proof of $\Gamma \vdash \Delta$ can be transformed into a proof of $\Gamma \vdash \Delta$ that does not contain Cut.

Sequent calculus for predicate logic

The quantifier rules are

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta}\, L\forall \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x.A, \Delta}\, R\forall$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}\, L\exists \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x.A, \Delta}\, R\exists,$$

where in $R\forall$ and $L\exists$ it must hold that $x \notin FV(\Gamma, \Delta)$ and in $L\forall$ and $R\exists$ it must hold that t is free for x in A.

Exercise

Show how

- $L\forall$ can be used to express the ND rule $\forall e$;
- $L\exists$ can be used to express the ND rule $\exists e$.