The sequent calculus

Sequents

“Sequent” is another word for “syntactic entailment” (recall lecture on ND). That is, a sequent is of the form

$$\Gamma \vdash B,$$

where Γ is a list of formulæ $A_1, \ldots, A_n$, and B is a formula. By soundness and completeness (of ND), we have

$$\Gamma \vdash B \text{ iff } \Gamma \models B.$$

Multiple conclusions

We also briefly considered sequents with **multiple conclusions**, i.e. of the form

$$\Gamma \vdash \Delta,$$

where Γ is a list of formulæ $A_1, \ldots, A_n$ and Δ is a list of formulæ $B_1, \ldots, B_m$. The intended meaning is

$$A_1 \wedge \ldots \wedge A_n \models B_1 \vee \ldots \vee B_m.$$

Towards sequent calculus

As we have seen, the natural-deduction calculus has an introduction rules and elimination rules for every connective, e.g.

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \wedge A_2} \wedge i$$

$$\frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_1} \wedge e \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_2} \wedge e.$$

Notice that all of the action happens on the right side.

The sequent calculus

- In his seminal 1934 paper, along with natural deduction, Gentzen also proposed an alternative to ND: the **sequent calculus**.
- Instead of the elimination rules, the sequent calculus has **left introduction rules**:

Rules for $\wedge$ and $\vee$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\, L\wedge \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\, R\wedge$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}\, L\vee \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\, R\vee$$

Note the pretty symmetry: $L\vee$ is the dual of $R\wedge$, and $L\wedge$ is the dual of $R\vee$.

The Axiom rule

We also have axioms of the form

$$\frac{}{A \vdash A}\,Ax.$$

True and False

The rules for $\top$ (true) and $\perp$ (false) are

$$\frac{}{\perp \vdash} L\perp \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta} R\perp$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta} L\top \qquad \frac{}{\vdash \top} R\top.$$

Implication

The rules for implication are

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'} L\to \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash A \to B, \Delta} R\to .$$

Negation and implication

As in the case of natural deduction, we define

$$\neg A = (A \rightarrow \bot).$$

This means that the following rules are derivable:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\, L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\, R\neg.$$

Exercise

In fact, we could have defined

$$A \to B = (\neg A \vee B).$$

Then we could derive the rules $L \to$ and $R \to$ from $L\neg$ and $R\neg$. Show this.

The structural rules

The introduction rules for the logical connectives are called “logical rules”. Besides those and the axiom rule, there is another essential set of rules: the **structural rules**.

- Exchange:

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,LE \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,RE$$

Structural rules

- Weakening:

$$\frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta}\, LW \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\, RW$$

- Contraction:

$$\frac{\Gamma, A, A, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta}\, LC \qquad \frac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\, RC$$

Significance of structural rules

- The structural rules correspond to the fact that contexts (which by definition are list of formulæ) can be seen as **sets**.
- We could have introduced contexts as sets from the beginning; but that would be unwise, because sometimes one **wants** contexts to be lists (e.g. in **linear logic, which is beyond the scope of this lecture**).

Using the structural rules

The structural rules allow us to simplify some other rules. E.g. consider

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma', \vdash A \wedge B, \Delta, \Delta'} R\wedge .$$

Because of LW and RW, the rule below suffices:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} R\wedge .$$

Rules of the first kind are called **multiplicative**, and rules of the second kind are called **additive**.

Exercise

Show that the multiplicative version together with the structural rules implies the additive version. Which structural rules are needed for that?

The Cut rule

The final rule of the sequent calculus is the famous **Cut**:

$$\frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3} \ Cut.$$

A is called the “cut formula”. As we shall see shortly, the Cut rule plays a key rôle in the translation of natural-deduction proofs into proofs of the sequent calculus.

Summary: Ax, Cut, logical rules

$$\frac{}{A \vdash A}\,Ax \qquad \frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,Cut$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,L\wedge \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma', \vdash A \wedge B, \Delta, \Delta'}\,R\wedge$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}\,L\vee \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,R\vee$$

$$\frac{}{\bot \vdash}\,L\bot \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\,R\bot$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta}\,L\top \qquad \frac{}{\vdash \top}\,R\top$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \rightarrow B \vdash \Delta, \Delta'}\,L\rightarrow \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash A \rightarrow B, \Delta}\,R\rightarrow$$

Summary: structural rules

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\, LE \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\, RE$$

$$\frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta}\, LW \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\, RW$$

$$\frac{\Gamma, A, A, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta}\, LC \qquad \frac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\, RC$$

Terminology

- The occurrences of Γ and Δ in the inference rules are called the **side formulæ** or the **context**.
- In the conclusion of each rule, the formula not in the context is called the **main** formula or **principal** formula. In the rule Ax, both occurrences of A are principal.
- The formula(s) in the premise(s) from which the principle formula derives are called the **active** formulas.