# Write essay. Choice of five topics. Submit by 16:15 on May 13 in Departmental Office (1 West 2.23). Proportion of unit assessment: 25%. Individual coursework, i.e. complete it on your own.

## **Five topics**

Undecidability of predicate logic.
 Explained later in this lecture.

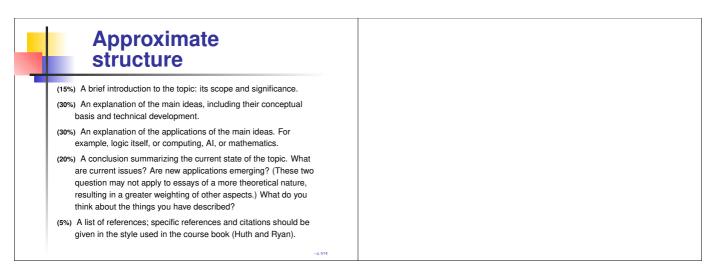
• Verification by model checking. This is about scenarios where we want to check automatically if  $M \models A$  for some reactive or concurrent system M and some modal or temporal formula A describing a desirable property.

### **Five topics**

- Compactness, and Löwenheim-Skolem theorems. This is about the connection between sets Γ of formulæ and their models.
- Gödel's incompleteness theorems. Famous results about the limits of deduction in predicate logic.
- Fuzzy logic and its applications. Reasoning about "partial truth". (Fairly easy subject; compensate by deep literature search.)

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Reference on writing essays Michael Alley, The Craft of Scientific Writing, Prentice-Hall, 1987.



# Summary of quantifier rules

The introduction and elimination rules for quantifiers are

 $\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A} \, \forall i \quad \text{if} \; x \not \in FV(\Gamma)$ 

 $\frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]} \, \forall e$ 

 $\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash A} \, \exists i$ 

$$\frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \exists e \quad \text{if } x \notin FV(\Gamma \cup \{B\}),$$

where for  $\forall e \text{ and } \exists i$ , the term t must be free for x in A.

### Soundness

(This slide and the next three are an improved version of the slides about soundness in the previous handout.) **Theorem.**[Soundness] If  $\Gamma \vdash A$ , then  $\Gamma \models A$ .

- The soundness of the rules for ∧, →, ⊥, and ∨ is shown in the same way as for propositional logic.
- So it remains to show the soundness of  $\forall i$ ,  $\forall e, \exists i$ , and  $\exists e$ .

- p.

#### **Soundness of** $\forall i$ and $\exists e$ The soundness proof for $\forall i$ works as follows: suppose that $\Gamma \models A$ and $M \models \Gamma$ . To see that $M \models \forall x.A$ , we need to show that $M[a/x] \models A$ for all $a \in U$ . Because $M \models \Gamma$ and x does not occur freely in $\Gamma$ , we have $M[a/x] \models \Gamma$ . Because $\Gamma \models A$ , we get $M[a/x] \models A$ . **Exercise:** Prove the soundness of $\exists e$ .

## Soundness of $\forall e$ and $\exists i$

The soundness proof for these two rules requires the following lemma, which can be proved by induction on A.

**Lemma.** For every formula A, every term t which is free for x in A, and every situation M, it holds that

 $M \models A[t/x]$  iff  $M[\llbracket t \rrbracket_M/x] \models A$ .

## Soundness of $\forall e$ and $\exists i$

The soundness proof for  $\forall e$  works as follows: suppose that  $\Gamma \models \forall x.A$ , and let t be free for x in A. To see that  $\Gamma \models A$ , suppose that  $M \models \Gamma$ . Because  $\Gamma \models \forall x.A$ , we have  $M[a/x] \models A$  for all  $a \in U$ . In particular,  $M[[t]]_M/x] \models A$ . By the lemma, this is so iff  $M \models A[t/x]$ .

**Exercise:** Prove the soundness of  $\exists i$ .

## Completeness

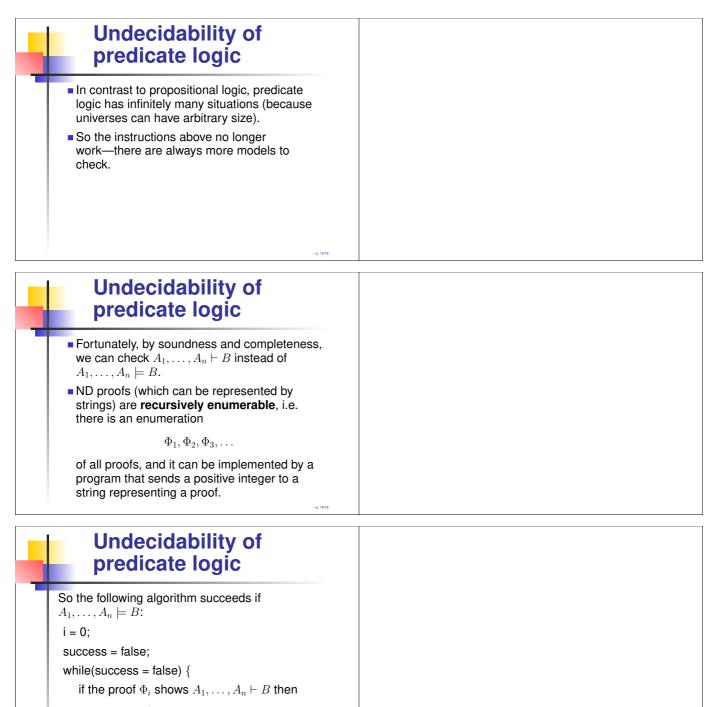
**Theorem.**[Completeness] If  $\Gamma \models A$ , then  $\Gamma \vdash A$  is provable in ND.

- The completeness proof follows the same scheme as the one for propositional logic.
- Only the Model Existence Lemma needs to be re-proved, because situations now involve a universe, functions, and predicates.
- The proof of the MEL is still based on (an updated version of) maximally consistent sets. (For details, see van Dalen.)

# Undecidability of predicate logic

Recall that the algorithm below can be used to decide the validity of  $A_1, \ldots, A_n \models B$  in propositional logic.

- 1. Check for every situation M if, whenever  $M \models A_i$  for all  $i \in \{1, ..., n\}$ , then  $M \models B$ .
- 2. If this is true, then  $A_1, \ldots, A_n \models B$ ,
- **3.** otherwise  $A_1, \ldots, A_n \not\models B$ .



success = true;

i = i+1;

}

# Undecidability of predicate logic

- The algorithm gives a positive answer if  $A_1, \ldots, A_n \models B$ .
- But goes into in infinite loop if  $A_1, \ldots, A_n \not\models B$ .
- Can this be fixed, i.e. is there an algorithm that also comes back with an answer if  $A_1, \ldots, A_n \not\models B$ ?
- The following theorem states that this is impossible.

# Undecidability of predicate logic

**Theorem.** There is no algorithm that, given any formula A of predicate logic, decides whether A is valid or not.

There are various proofs of the undecidability theorem (see e.g. Huth/Ryan, Boolos/Burgess/Jeffrey). This is one of the possible coursework essays.

**First-order logic** 

Predicate logic is also called **first-order logic**. This terminology refers to the types of the variables. For example,

 $\exists f. \forall x. f(x) = x$ 

is a **second-order formula**, because f ranges over functions  $U \to U$ , not elements of U. A formula is third-order if it contains quantifies ranging over things of type  $(U \to U) \to U$ , and so on.