Natural deduction for predicate logic

ND for predicate logic

The rules of ND for predicate logic are those of ND for propositional logic, plus introduction rules and elimination rules for $\forall$ and $\exists$.

$\forall$-elimination, first attempt

The rule for $\forall$-elimination is as follows, where t can be any term, and $[t/x]$ means that t is substituted for every free occurrence of x in A. (We shall formalize soon what "free" means.)

$$\frac{\forall x.A}{A[t/x]}\ \forall e$$

This is intuitively clear—consider

$$\frac{\text{for all numbers } n \text{ it holds that } n \text{ is even or } n \text{ is odd}}{9 \text{ is even or } 9 \text{ is odd}}\ .$$

But there is a catch. . .

Variable capture

- Consider e.g. the formula below, which holds e.g. for the natural numbers.

$$A = \forall x . \exists y . x < y$$

- Applying $\forall$-elimination with $t = y$ yields the following formula, which is not valid.

$$\exists y . y < y$$

- The mistake has been caused by **variable capture**: the variable y in t has been caught by the quantifier $\exists y$.

Scope

To make precise what variable capture is, we define the notion of **scope**.

Definition. The **scope** of the occurrence of a quantifier $\forall x$ or $\exists x$ in a formula A is obtained as follows:

1. Let $\forall x.B$ be the subformula of A that starts with the above quantifier occurrence.
2. Remove all subformulæ of B that also start with a quantifier for x ($\forall$ or $\exists$).

Scope: example

Example. The scope of the right-hand $\forall x$ in the formula

$$(\forall x.p(x)) \wedge \forall x.(p(x) \rightarrow \exists x.q(x))$$

is $p(x) \rightarrow \bullet$, where $\bullet$ stands for the hole that results from removing $\exists x.q(x)$.

Free variable occurrences

Another definition we need to address the issue of variable capture:

Definition. An occurrence of a variable x in a formula A is said to be **free** if it is neither part of a quantifier ($\forall x$ or $\exists x$) nor in the scope of a quantifier for x.

Example. The left x is free in the formula below, while the other two are not.

$$p(x) \wedge \forall x . p(x)$$

Avoiding variable capture

Next, we define the notion we shall use to avoid variable capture:

Definition. Given a term t, a variable x and a formula A, we say that t is **free for** x **in** A if A has no free occurrence of x in the scope of a quantifier $\forall y$ or $\exists y$ for any variable y occurring in t. (In other words, if no variable capture happens during the substitution $A[t/x]$.)

$\forall$-elimination, final version

In the style without assumptions:

$$\frac{\forall x.A}{A[t/x]}\,\forall e \quad \text{if } t \text{ is free for } x \text{ in } A$$

In the style with assumptions:

$$\frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]}\,\forall e \quad \text{if } t \text{ is free for } x \text{ in } A$$

$\forall$-introduction

In the style with assumptions, the rule for $\forall$-introduction is

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}\,\forall i \quad \text{if } x \notin FV(\Gamma).$$

Intuitively,

$$\frac{A \text{ holds of an arbitrary } x}{A \text{ holds for all } x}\;.$$

From a syntactic point of view, “arbitrary” means that x is not used in the assumptions.

$\forall$-introduction

The rule for $\forall$-introduction in the style without assumptions is

$$\frac{A}{\forall x.A}\,\forall i$$

if no **undischarged** assumption of A has a free occurrence of x.

Natural deduction: example

Assuming that x does not occur freely in A, we have the following ND proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x.(A \rightarrow B)]_2}{A \rightarrow B}\,\forall e \qquad [A]_1}{B}\,\rightarrow e}{\forall x.B}\,\forall i}{A \rightarrow \forall x.B}\,\rightarrow i_1}{(\forall x.(A \rightarrow B)) \rightarrow (A \rightarrow \forall x.B)}\,\rightarrow i_2.$$

The side condition for the $\forall$-elimination is "x is free for x in $A \rightarrow B$". **Exercise:** show that x is free for x in any formula.

Exercises

Show:

1. $\vdash (\forall x.(A(x) \wedge B(x))) \leftrightarrow ((\forall x.A(x)) \wedge (\forall x.B(x)))$.
2. $\vdash (\forall x.(A(x) \rightarrow B(x))) \rightarrow ((\forall x.A(x)) \rightarrow (\forall x.B(x)))$. (Which condition is required for the converse? Explain!)
3. $\vdash A \leftrightarrow \forall x.A$ where $x \notin FV(A)$.
4. $\vdash (\forall x.A(x)) \rightarrow \neg\forall x.\neg A(x)$.
5. $\vdash (\forall x.\forall y.A(x,y)) \rightarrow \forall x.A(x,x)$. (Does this require a side condition? Explain!)

$\exists$-introduction

$$\frac{A[t/x]}{\exists x.A}\,\exists i \quad \text{if } t \text{ is free for } x \text{ in } A$$

The intuition is almost trivial:

$$\frac{A(x) \text{ holds for some witness } t \text{ instead of } x}{\text{there exists some } x \text{ such that } A(x) \text{ holds}}\,.$$

- The side condition only makes sure that t contains no variables in the scope of quantifiers.

$\exists$-elimination

In the style with explicit assumptions, the rule for $\exists$-elimination is

$$\frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,\exists e \quad x \notin FV(\Gamma \cup \{B\}).$$

Intuitively,

$$\frac{\begin{array}{c}\text{there is an } x \text{ such that } A(x) \\ \text{an arbitrary } x \text{ s.t. } A(x) \text{ implies } B\end{array}}{B \text{ holds}}\,.$$

Technically, “arbitrary” means that neither the assumptions nor the conclusion B contain x.

$\exists$-elimination

In the style without explicit assumptions, the rule for $\exists$-elimination is

$$\dfrac{\exists x.A(x) \qquad \begin{matrix}[A(x)] \\ \vdots \\ B\end{matrix}}{B}\ \exists e$$

if neither the undischarged assumptions nor B have free occurrences of x.

Note the similarity with $\vee e$.

Example

$$
\dfrac{\dfrac{\dfrac{[\exists x.\neg A]_2 \qquad \dfrac{[\neg A]_1 \qquad \dfrac{[\forall x.A]_3}{A}\,\forall e}{\bot}\,\to e}{\bot}\,\exists e_1}{\neg\exists x.\neg A}\,\to i_2}{\forall x.A \to \neg\exists x.\neg A}\,\to i_3
$$

Example

The following proof shows the converse of the formula proved on the previous slide.

$$\cfrac{\cfrac{\cfrac{\cfrac{[\neg\exists x.\neg A]_2 \qquad \cfrac{[\neg A]_1}{\exists x.\neg A}\,\exists i}{\bot}\,\rightarrow e}{A}\,RAA_1}{\forall x.A}\,\forall i}{\neg\exists x.\neg A \rightarrow \forall x.A}\,\rightarrow i_2$$

Note that this proof uses RAA. The formula $\neg\exists x.\neg A \rightarrow \forall x.A$ does not hold in intuitionistic logic.

Exercise

Show that $\exists$ can be expressed in terms of $\forall$ by defining

$$\exists x.A = \neg\forall x.\neg A,$$

in the sense that the introduction and elimination rules for $\exists$ follow from the other rules of ND.

Exercise

Show the claims below, where $x \notin FV(B)$.

1. $\vdash (\forall x.(A(x) \to B)) \to ((\exists x.A(x)) \to B)$.
2. $\vdash \exists x.(A(x) \vee B(x)) \to ((\exists x.A(x)) \vee (\exists x.B(x)))$.
3. $\vdash (\exists x.(A(x) \wedge B)) \leftrightarrow ((\exists x.A(x)) \wedge B)$.
4. $\vdash (\forall x.(A(x) \vee B)) \leftrightarrow ((\forall x.A(x)) \vee B)$.
5. $\vdash (\exists x.A(x)) \leftrightarrow \neg\forall x.\neg A(x)$.

(Some of these are hard—do not worry if you cannot solve all five exercises.)

Summary of quantifier rules

The introduction and elimination rules for quantifiers are

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}\,\forall i \quad \text{if } x \notin FV(\Gamma) \qquad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]}\,\forall e$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash A}\,\exists i \quad \frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,\exists e \quad x \notin FV(\Gamma \cup \{A\}),$$

where for $\forall e$ and $\exists i$, the term t must be free for x in A.

Soundness

Theorem.[Soundness] If $\Gamma \vdash A$, then $\Gamma \models A$.

- The soundness of the rules for $\wedge$, $\longrightarrow$, $\perp$, and $\vee$ is shown in the same way as for propositional logic.
- Showing the soundness of $\forall i$, $\forall e$, $\exists i$, and $\exists e$ is fairly easy.

Exercise

The soundness proof for $\forall i$ works as follows: suppose that $\Gamma \models A$ and $M \models \Gamma$. To see that $M \models \forall x.A$, we need to show that $M[a/x] \models A$ for all $a \in U$. Because $M \models \Gamma$ and x does not occur freely in Γ, we have $M[a/x] \models \Gamma$. Because $\Gamma \models A$, we get $M[a/x] \models A$.

Exercise: Prove the soundness of the remaining quantifier rules.

Completeness

Theorem.[Completeness] If $\Gamma \models A$, then $\Gamma \vdash A$.

- The completeness proof follows the same scheme as the one for propositional logic.
- Only the Model Existence Lemma needs to be re-proved, because situations now involve a universe, functions, and predicates.
- While the proof of Model Existence Lemma is still based on (an updated version of) maximally consistent sets, it is much harder than in the propositional case.