## Natural deduction for predicate logic

# ND for predicate logic

The rules of ND for predicate logic are those of ND for propositional logic, plus introduction rules and elimination rules for  $\forall$  and  $\exists$ .

## ∀-elimination, first attempt

The rule for  $\forall$ -elimination is as follows, where t can be any term, and [t/x] means that t is substituted for every free occurrence of x in A. (We shall formalize soon what "free" means.)

$$\frac{\forall x.A}{A[t/x]} \,\forall e$$

This is intuitively clear—consider

for all numbers n it holds that n is even or n is odd 9 is even or 9 is odd

But there is a catch...

## Variable capture

Consider e.g. the formula below, which holds e.g. for the natural numbers.

$$A = \forall x. \exists y. x < y$$

Applying  $\forall$ -elimination with t = y yields the following formula, which is not valid.

$$\exists y. y < y$$

The mistake has been caused by variable capture: the variable *y* in *t* has been caught by the quantifier ∃*y*.



To make precise what variable capture is, we define the notion of **scope**.

**Definition.** The **scope** of the occurrence of a quantifier  $\forall x$  or  $\exists x$  in a formula A is obtained as follows:

- 1. Let  $\forall x.B$  be the subformula of A that starts with the above quantifier occurrence.
- 2. Remove all subformulæ of *B* that also start with a quantifier for  $x \ (\forall \text{ or } \exists)$ .

## Scope: example

**Example.** The scope of the right-hand  $\forall x$  in the formula

$$(\forall x.p(x)) \land \forall x.(p(x) \to \exists x.q(x)))$$

is  $p(x) \rightarrow \bullet$ , where  $\bullet$  stands for the hole that results from removing  $\exists x.q(x)$ .

## Free variable occurrences

Another definition we need to address the issue of variable capture:

**Definition.** An occurrence of a variable x in a formula A is said to be **free** if it is neither part of a quantifier ( $\forall x \text{ or } \exists x$ ) nor in the scope of a quantifier for x.

**Example.** The left x is free in the formula below, while the other two are not.

 $p(x) \land \forall x. p(x)$ 

# Avoiding variable capture

Next, we define the notion we shall use to avoid variable capture:

**Definition.** Given a term t, a variable x and a formula A, we say that t is **free for** x **in** A if A has no free occurrence of x in the scope of a quantifier  $\forall y$  or  $\exists y$  for any variable y occurring in t. (In other words, if no variable capture happens during the substitution A[t/x].)

## ∀-elimination, final version

In the style without assumptions:

 $\frac{\forall x.A}{A[t/x]} \,\forall e \quad \text{if } t \text{ is free for } x \text{ in } A$ 

In the style with assumptions:

$$\frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]} \,\forall e \quad \text{if } t \text{ is free for } x \text{ in } A$$

## ∀-introduction

In the style with assumptions, the rule for  $\forall$ -introduction is

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A} \,\forall i \quad \text{if } x \notin FV(\Gamma).$$

Intuitively,

## $\frac{A \text{ holds of an arbitrary } x}{A \text{ holds for all } x}$

From a syntactic point of view, "arbitrary" means that x is not used in the assumptions.

## ∀-introduction

The rule for  $\forall$ -introduction in the style without assumptions is

 $\frac{A}{\forall x.A} \forall i$  if no **undischarged** assumption of *A* has a free occurrence of *x*.

## Natural deduction: example

Assuming that x does not occur freely in A, we have the following ND proof:

$$\frac{ \begin{bmatrix} \forall x.(A \to B) \end{bmatrix}_2}{A \to B} \forall e \quad [A]_1 \to e \\ \frac{B}{\forall x.B} \forall i \\ \frac{A \to \forall x.B}{\forall x.B} \to i_1 \\ (\forall x.(A \to B)) \to (A \to \forall x.B) \to i_2.$$

The side condition for the  $\forall$ -elimination is "x is free for x in  $A \rightarrow B$ ". **Exercise:** show that x is free for x in any formula.

## **Exercises**

#### Show:

- $1. \vdash (\forall x.(A(x) \land B(x))) \leftrightarrow ((\forall x.A(x)) \land (\forall x.B(x))).$
- 2.  $\vdash (\forall x.(A(x) \rightarrow B(x))) \rightarrow ((\forall x.A(x)) \rightarrow (\forall x.B(x)))$ . (Which condition is required for the converse? Explain!)
- **3.**  $\vdash A \leftrightarrow \forall x.A$  where  $x \notin FV(A)$ .
- **4.**  $\vdash (\forall x.A(x)) \rightarrow \neg \forall x.\neg A(x).$
- 5.  $\vdash (\forall x.\forall y.A(x,y)) \rightarrow \forall x.A(x,x)$ . (Does this require a side condition? Explain!)

## **∃-introduction**

$$\frac{A[t/x]}{\exists x.A} \exists i \quad \text{if } t \text{ is free for } x \text{ in } A$$

The intuition is almost trivial:

A(x) holds for some witness t instead of x there exists some x such that A(x) holds

The side condition only makes sure that t contains no variables in the scope of quantifiers.

## **∃-elimination**

In the style with explicit assumptions, the rule for  $\exists$ -elimination is

$$\frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \exists e \quad x \notin FV(\Gamma \cup \{B\}).$$

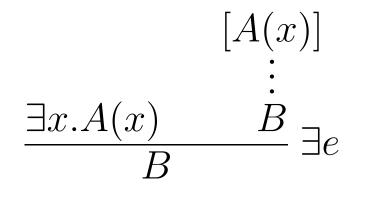
Intuitively,

there is an x such that A(x)an arbitrary x s.t. A(x) implies BB holds

Technically, "arbitrary" means that neither the assumptions nor the conclusion B contain x.

## **]-elimination**

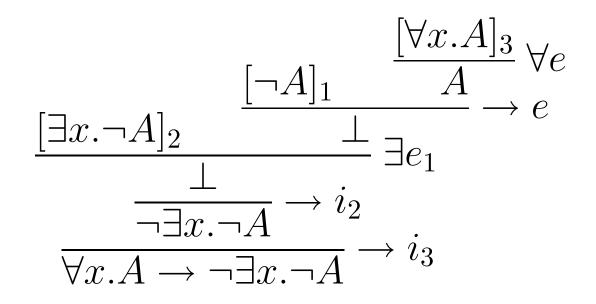
In the style without explicit assumptions, the rule for  $\exists$ -elimination is



if neither the undischarged assumptions nor B have free occurrences of x.

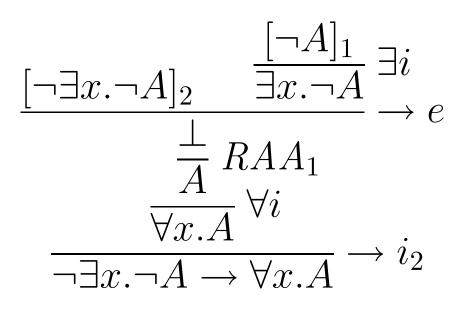
Note the similarity with  $\lor e$ .







The following proof shows the converse of the formula proved on the previous slide.



Note that this proof uses RAA. The formula  $\neg \exists x. \neg A \rightarrow \forall x. A$  does not hold in intuitionistic logic.

### Exercise

Show that  $\exists$  can be expressed in terms of  $\forall$  by defining

#### $\exists x.A = \neg \forall x. \neg A,$

in the sense that the introduction and elimination rules for  $\exists$  follow from the other rules of ND.

### Exercise

Show the claims below, where  $x \notin FV(B)$ . 1.  $\vdash (\forall x.(A(x) \rightarrow B)) \rightarrow ((\exists x.A(x)) \rightarrow B).$ **2.**  $\vdash \exists x.(A(x) \lor B(x)) \rightarrow ((\exists x.A(x)) \lor (\exists x.B(x))).$ **3.**  $\vdash (\exists x.(A(x) \land B)) \leftrightarrow ((\exists x.A(x)) \land B).$ **4.**  $\vdash (\forall x.(A(x) \lor B)) \leftrightarrow ((\forall x.A(x)) \lor B).$ **5.**  $\vdash (\exists x.A(x)) \leftrightarrow \neg \forall x.\neg A(x).$ (Some of these are hard—do not worry if you cannot solve all five exercises.)

## Summary of quantifier rules

The introduction and elimination rules for quantifiers are

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A} \,\forall i \quad \text{ if } x \not\in FV(\Gamma) \qquad \quad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]} \,\forall e$$

 $\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash A} \exists i \quad \frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \exists e \quad x \not\in FV(\Gamma \cup \{A\}),$ 

where for  $\forall e \text{ and } \exists i$ , the term t must be free for x in A.

## Soundness

**Theorem.**[Soundness] If  $\Gamma \vdash A$ , then  $\Gamma \models A$ .

- The soundness of the rules for ∧, →, ⊥, and ∨ is shown in the same way as for propositional logic.
- Showing the soundness of  $\forall i, \forall e, \exists i, and \exists e is fairly easy.$

### Exercise

The soundness proof for  $\forall i$  works as follows: suppose that  $\Gamma \models A$  and  $M \models \Gamma$ . To see that  $M \models \forall x.A$ , we need to show that  $M[a/x] \models A$  for all  $a \in U$ . Because  $M \models \Gamma$  and x does not occur freely in  $\Gamma$ , we have  $M[a/x] \models \Gamma$ . Because  $\Gamma \models A$ , we get  $M[a/x] \models A$ .

**Exercise:** Prove the soundness of the remaining quantifier rules.

## Completeness

**Theorem.**[Completeness] If  $\Gamma \models A$ , then  $\Gamma \vdash A$ .

- The completeness proof follows the same scheme as the one for propositional logic.
- Only the Model Existence Lemma needs to be re-proved, because situations now involve a universe, functions, and predicates.
- While the proof of Model Existence Lemma is still based on (an updated version of) maximally consistent sets, it is much harder than in the propositional case.