Soundness and completeness

Soundness and completeness

We want the syntactic entailment $\vdash$ and the semantic entailment $\models$ to agree. This requirement consist of two parts:

Soundness: If $\Gamma \vdash A$ is provable in ND, then $\Gamma \models A$.

Completeness: If $\Gamma \models A$, then $\Gamma \vdash A$ is provable in ND.

Soundness and completeness

- Soundness an completeness are key requirements of any logic.
- We shall now turn to proving soundness and completeness for propositional logic.
- This is important, because tweaking the following proofs yields soundness and completeness results of more sophisticated logics later in this course.

Soundness

Proposition.[Soundness] If $\Gamma \vdash A$ is derivable in the natural-deduction calculus, then it holds that $\Gamma \models A$.

We show this by induction on the size of the natural-deduction proof.

Recall this exercise

Exercise from earlier lecture: prove the following facts about semantic entailment.

$$\frac{\Gamma \models A \quad \Gamma \models B}{\Gamma \models A \wedge B} \wedge i \qquad \frac{\Gamma \models A \wedge B}{\Gamma \models A} \wedge e \qquad \frac{\Gamma \models A \wedge B}{\Gamma \models B} \wedge e$$

$$\frac{\Gamma, A \models B}{\Gamma \models A \rightarrow B} \rightarrow i \qquad \frac{\Gamma \models A \rightarrow B \quad \Gamma \models A}{\Gamma \models B} \rightarrow e$$

$$\frac{\Gamma \models \bot}{\Gamma \models A} \bot e \qquad \frac{\Gamma, \neg A \models \bot}{\Gamma \models A} RAA \qquad \frac{}{\Gamma, A \models A} Ax$$

These are all the facts needed for the soundness proof.

Soundness proof: base case

The base case of the induction is given by the smallest proofs; they are of the form

$$\frac{}{\Gamma, A \vdash A}\, Ax.$$

We need to show that

$$\Gamma, A \models A.$$

But this is trivial: every model of $\Gamma \cup \{A\}$ is in particular a model of A.

Soundness proof: $\wedge$-introduction

Suppose that we have a natural-deduction proof of

$$\Gamma \vdash A \wedge B.$$

with last rule $\wedge i$. Then we have shorter proofs of

$$\Gamma \vdash A \quad \text{and} \quad \Gamma \vdash B.$$

By induction hypothesis, we have

$$\Gamma \models A \quad \text{and} \quad \Gamma \models B.$$

This evidently implies $\Gamma \models A \wedge B$.

Soundness proof: $\longrightarrow$-introduction

Suppose that we have a natural-deduction proof of

$$\Gamma \vdash A \to B.$$

with last rule $\to i$. Then we have a shorter proof of

$$\Gamma, A \vdash B.$$

By induction hypothesis, we have

$$\Gamma, A \models B.$$

As is easy to see, this implies $\Gamma \models A \to B$.

Soundness of the remaining rules

- Recall that I have shown the soundness of $\rightarrow e$ and RAA earlier!
- Soundness of $\bot e$: trivial exercise.
- The soundness proof is also in van Dalen.

This concludes the proof of the soundness proposition.

Completeness

Theorem.[Completeness] If $\Gamma \models A$, then $\Gamma \vdash A$ is provable in ND.

As with most logics, the completeness of propositional logic is harder (and more interesting) to show than the soundness. We shall spend the next few slides with the completeness proof.

Completeness proof: consistency

The notion of **consistency** plays a key rôle in the completeness proof.

Definition. A set Γ of formulæ is called **consistent** if $\Gamma \nvdash \bot$.

In other words, Γ is consistent if it does not allow the proof of a contradiction.

Completeness proof

We prove completeness by showing the **contrapositive**:

$$\Gamma \nvdash A \quad \text{implies} \quad \Gamma \not\models A:$$

$\Gamma \nvdash A$	implies $\Gamma \cup \{\neg A\}$ is consistent	(easy)
	implies $\Gamma \cup \{\neg A\}$ has a model	(Model Existence Lemma)
	implies $\Gamma \not\models A$	(obvious)

The Model Existence Lemma is the centerpiece:

Model Existence Lemma: Every consistent set of formulæ has a model.

Maximally consistent sets

Maximally consistent sets play a key rôle in the completeness proof.

Definition. A set Γ of formulæ is called **maximally consistent** is it is consistent and adding any further formulæ makes it inconsistent.
The proof of the Model Existence Lemma use the fact that that every maximally consistent set Γ^* has a model.

About maximally consistent sets

Lemma. Every consistent set Γ is contained in a maximally consistent set Γ^*.

Proof. See blackboard or van Dalen.

About maximally consistent sets

Lemma. Maximally closed sets are closed under provability, i.e. if Γ is maximally consistent, then

$$\Gamma \vdash A \quad \text{implies} \quad A \in \Gamma$$

Proof.(No need to remember this.) Suppose that $\Gamma \vdash A$, but $A \notin \Gamma$. Because Γ is maximally consistent, $\Gamma \cup \{A\}$ must be inconsistent, i.e. $\Gamma, A \vdash \perp$. By $\rightarrow i$, we have $\Gamma \vdash A \rightarrow \perp$. By $\rightarrow e$, we get $\Gamma \vdash \perp$, i.e. Γ is inconsistent. Contradiction!

About maximally consistent sets

Lemma. If Γ is maximally consistent, then

$$(A \to B) \in \Gamma \quad \text{iff} \quad (A \in \Gamma \quad \text{implies} \quad B \in \Gamma).$$

Proof.(No need to remember this.) Left-to-right: suppose that $(A \to B) \in \Gamma$ and $A \in \Gamma$. By modus ponens, we get $\Gamma \vdash B$. By an earlier lemma, Γ is closed under deduction, so $B \in \Gamma$.

Right-to-left. Suppose that $A \in \Gamma$ implies $B \in \Gamma$. To see that $(A \to B) \in \Gamma$, we consider two cases. Case 1: $A \in \Gamma$. Then by assumption we have $B \in \Gamma$, so $\Gamma = \Gamma \cup \{A\} \vdash B$. By the $\to i$ rule, we get $\Gamma \vdash A \to B$. Because Γ is closed under deduction, we have $(A \to B) \in \Gamma$. Case 2: $A \notin \Gamma$. Because Γ is maximally consistent, we have $\Gamma, A \vdash \bot$. By the $\bot e$ rule, we have $\Gamma, A \vdash B$. By $\to i$, we have $\Gamma \vdash A \to B$.

About maximally consistent sets

Note that letting $B = \perp$ in the previous lemma yields

Lemma. If Γ is maximally consistent, then

$$\neg A \in \Gamma \quad \text{iff} \quad A \notin \Gamma.$$

The Model Existence Lemma

To prove completeness, it remains to prove the Model Existence Lemma.

Lemma. Every consistent set Γ of formulæ has a model.

Proof. Blackboard or van Dalen.

This concludes the completeness proof for propositional logic.

What to remember

What to remember about the completeness proof:

- The overview, which states that we prove the contrapositive in three steps.
- What the Model Existence Lemma states and how that statement is used in the completeness proof.
- What consistent and maximally consistent sets are.
- That every consistent set is contained in a maximally consistent set and why.
- That the proof of the MEL works basically by constructing a model from a maximally consistent set.