Natural deduction

Motivation for formal inference systems

How can we check if

$$A_1, \ldots, A_n \models B?$$

For propositional logic, there is an algorithm:

1. Check for every situation M if, whenever $M \models A_i$ for all $i \in \{1, \ldots, n\}$, then $M \models B$.
2. If this is true, then $A_1, \ldots, A_n \models B$,
3. Otherwise $A_1, \ldots, A_n \not\models B$.

Why does this algorithm always terminate?

Motivation for formal inference systems

- The algorithm terminates because there are only finitely many situations:
- Let $p_1, \ldots, p_m$ be the propositional atoms that occur in $\{A_1, \ldots, A_n, B\}$.
- A situation corresponds to a truth table, e.g.

p_1	p_2	p_3	$\ldots$	p_m
0	1	1	$\ldots$	0

.

- There are 2 possibilities for each p_i, so the number of situations we have to try is 2^m.

Motivation for formal inference systems

- However, there are other logics (e.g. predicate logic) with infinitely many situations.
- So the method we have just seen can no longer terminate with a positive result (because there are always more situations to check).
- So we need a different way of showing $\Gamma \models A$.
- The first such system we shall study is **natural deduction**.

Another such system was presented in Dan Richardson's second-year lecture: tableaux.

Natural deduction

- Natural deduction is called so because it mimics human reasoning in real life (in particular, in maths).
- ND systems exist for various logics (propositional logic, predicate logic, modal logic, intuitionistic logic...)
- We begin with ND for propositional logic because it is the simplest.
- We shall see ND systems for more sophisticated logics later.

Natural deduction

Natural deduction is a calculus to derive entailments

$$\Gamma \models A$$

step by step, in a purely symbolic way, without referring to situations.

$\wedge$-introduction

- If $\Gamma \models A$ and $\Gamma \models B$, then evidently $\Gamma \models A \wedge B$.
- To account for this, ND has the rule

$$\frac{A \quad B}{A \wedge B} \wedge i$$

- $\wedge i$ is the name of the rule; its stands for "and-introduction".
- The formulæ above the horizontal line are the **premises** of the rule.
- The formula below the line is the **conclusion**.

$\wedge$-elimination

- If $\Gamma \models A \wedge B$, then evidently $\Gamma \models A$ and $\Gamma \models B$.
- To account for this, the calculus has the rules

$$\frac{A \wedge B}{A}\ \wedge e \qquad \text{and} \qquad \frac{A \wedge B}{B}\ \wedge e.$$

A natural-deduction proof

The following is a proof of $p \wedge q, r \models p \wedge r$ in the ND calculus.

$$\cfrac{\cfrac{p \wedge q}{p}\,\wedge e \qquad r}{p \wedge r}\,\wedge i$$

- Note that the proof is a tree.
- The root is $p \wedge r$.
- The left branch leads to the leaf $p \wedge q$, via p.
- The right branch leads directly to the leaf r.

$\rightarrow$-elimination

- As we have seen, $\wedge$ has introduction and elimination rules.
- The same is true for every connective.
- $\rightarrow$-elimination is the aforementioned **modus ponens**:

$$\frac{A \quad A \rightarrow B}{B} \rightarrow e$$

Example:

$$\frac{\text{It rained} \quad \text{If it rained, then the street is wet}}{\text{The street is wet}}$$

A natural-deduction proof

The following proof shows $p, q, p \wedge q \to r \models r$.

$$\dfrac{\dfrac{p \quad q}{p \wedge q}\wedge i \qquad p \wedge q \to r}{r} \to e$$

- What is the root of this proof-tree?
- How many leaves has it got?

$\to$-introduction

- Evidently, if $\Gamma, A \models B$, then $\Gamma \models A \to B$.
- Note that A moves from the left to the right.
- Here is the $\to$-introduction rule:

$$\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \to B} \to i$$

- The square brackets mean that the assumption A is removed—the technical word is **discharged**.

A natural-deduction proof

The following proof shows
$p \wedge q \rightarrow r \models p \rightarrow (q \rightarrow r)$.

$$\dfrac{\dfrac{\dfrac{p \wedge q \rightarrow r \qquad \dfrac{[p]_2 \qquad [q]_1}{p \wedge q}\wedge i}{r} \rightarrow e}{q \rightarrow r} \rightarrow i_1}{p \rightarrow (q \rightarrow r)} \rightarrow i_2$$

The subscripts 1 and 2 indicate in which order the assumptions are discharged.

A natural-deduction proof

The following proof shows the converse of the entailment on the previous slide: $p \to (q \to r) \models p \wedge q \to r$.

$$
\dfrac{\dfrac{p \to (q \to r) \qquad \dfrac{[p \wedge q]_1}{p}\,\wedge e}{q \to r}\,\to e \qquad \dfrac{[p \wedge q]_1}{q}\,\wedge e}{\dfrac{r}{p \wedge q \to r}\,\to i_1}\,\to e
$$

The two superscripts 1 indicate that the two occurrences of $p \wedge q$ are considered the same, and are discharged simultaneously.

Negation

For reasons that will become clearer later, we define negation in terms of implication and falsity:

$$\neg A = (A \to \bot).$$

Note that this implies that the introduction and elimination rules for $\to$ apply in particular to $\neg$.

A natural-deduction proof

The following proof shows $A \models \neg\neg A$.

$$\dfrac{\dfrac{[A \to \bot]_1 \qquad A}{\bot} \to e}{(A \to \bot) \to \bot} \to i_1$$

Reductio ad absurdum (RAA)

- The converse of the entailment of the previous slide is $\neg\neg A \models A$.
- Evidently, it is valid with respect to the truth-table semantics.
- Remarkably, it is not provable with the rules shown so far.
- So we need to add it to the calculus.

Reductio ad absurdum (RAA)

The RAA rule is

$$\dfrac{\begin{array}{c}[\neg A] \\ \vdots \\ \bot\end{array}}{A}\ RAA.$$

- It is the only rule which is neither an introduction rule nor an elimination rule.
- The English name for this rule is **proof by contradiction**

Reductio ad absurdum (RAA)

- RAA is the only rule which is neither an introduction rule nor an elimination rule.
- It is considered invalid by constructivists. (We shall come back to this when we discuss intuitionistic logic.)
- But it is needed to prove all entailments that hold w.r.t. the truth-table semantics we are currently considering.

A natural-deduction proof

The following proof shows $\neg B \to \neg A \models A \to B$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg B \to \neg A \quad [\neg B]_1}{\neg A}\to e \qquad [A]_2}{\bot}\to e}{B}\,RAA_1}{A \to B}\to i_2
$$

Ex falso quodlibet

Finally, we need a rule that states that false entails anything (the Latin phrase is “ex falso [sequitur] quodlibet”).

$$\frac{\perp}{A}\,\perp e$$

This is an elimination rule (no introduction rule is needed for $\perp$).

Dropping $\vee$ and $\top$

- For the time being, we ignore the connectives $\vee$ and $\top$.
- This is no real loss, because they can be defined in terms of other connectives:

$$A \vee B = \neg(\neg A \wedge \neg B)$$
$$\top = (\bot \to \bot).$$

Summary of ND

Definition. A **natural deduction proof** is a finite tree whose leaves are formulæ (over the alphabet $\wedge, \rightarrow, \bot$) and which is built by using only the rules below.

$$\frac{A \quad B}{A \wedge B} \wedge i \qquad \frac{A \quad B}{A} \wedge e \qquad \frac{A \quad B}{B} \wedge e$$

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \rightarrow B} \rightarrow i \qquad \frac{A \rightarrow B \quad A}{B} \rightarrow e$$

$$\frac{\bot}{A} \bot e \qquad \frac{\begin{array}{c}\neg A \\ \vdots \\ \bot\end{array}}{A} RAA$$

Exercises

Prove the validity of the following semantic entailments by using natural deduction:

$$(A \wedge B) \wedge C \models A \wedge (B \wedge C)$$

$$\neg A \rightarrow \bot \models A$$

$$\models (A \wedge A) \rightarrow A$$

$$\bot \models A.$$

Exercises

Prove the validity of the following formulæ by using natural deduction:

$$(A \wedge B) \rightarrow (B \wedge A)$$
$$(\neg A \wedge A) \rightarrow \bot$$
$$A \rightarrow A \wedge A$$
$$((A \rightarrow B) \rightarrow A) \rightarrow A$$
$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)).$$

(Note that we have seen these laws before.)

Syntactic entailment

Definition. For a set of formula Γ and a formula A, we define

$$\Gamma \vdash A$$

if A follows from assumptions Γ in the natural-deduction calculus. We call the relation $\vdash$ **syntactic entailment**.

A different presentation of ND

There is an equivalent presentation of ND that defines $\vdash$ directly. Note that $\rightarrow i$ is the only rule where the assumptions change (because A is discharged).

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge i \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge e \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge e$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow e$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \bot e \qquad \frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A} RAA$$

A different presentation of ND

Actually, we need to add one rule to make this different presentation work:

$$\frac{}{\Gamma, A \vdash A}\ Ax$$

It only states the evident fact that assumptions can be used immediately as conclusions.

A different presentation of ND

Proofs with explicit assumptions have advantages, but the price is visual clutter: compare the proof

$$\dfrac{\dfrac{p \land q \rightarrow r \qquad \dfrac{[p]_2 \quad [q]_1}{p \land q}\land i}{\dfrac{r}{q \rightarrow r}\rightarrow i_1}\rightarrow e}{p \rightarrow (q \rightarrow r)}\rightarrow i_2$$

with the following proof of the same entailment. . .

A different presentation of ND

$$
\dfrac{
  \dfrac{
    \dfrac{}{p \wedge q \to r, p, q \vdash p \wedge q \to r}\,Ax
    \qquad
    \dfrac{
      \dfrac{}{p \wedge q \to r, p, q \vdash p}\,Ax
      \qquad
      \dfrac{}{p \wedge q \to r, p, q \vdash q}\,Ax
    }{p \wedge q \to r, p, q \vdash p \wedge q}\,\wedge i
  }{
    \dfrac{p \wedge q \to r, p, q \vdash r}{p \wedge q \to r, p \vdash q \to r}\,\to i
  }\,\to e
}{p \wedge q \to r \vdash p \to (q \to r)}\,\to i
$$