Propositional logic (revision) & semantic entailment

Reading

The background reading for propositional logic is Chapter 1 of Huth/Ryan. (This will cover approximately the first three lectures.)

Logical propositions

The basic building blocks of any logic are **logical formulæ** (also called "propositions" or "sentences").

Examples:

- Propositional logic: $p \wedge (p \rightarrow q) \rightarrow q$, $p \wedge \neg p$, $(p \wedge \neg q) \vee (q \wedge \neg p)$.
- Predicate logic: $\forall x. \exists y : f(x, g(y)) = c$.
- Modal logic: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.

The language of propositional logic

Definition. The language of propositional logic has an alphabet consisting of

- **propositional atoms**: $p, q, r, \ldots$
- **connectives**: $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\top$, $\bot$
- auxiliary symbols: (,)

The language of propositional logic

The connectives carry the traditional names:

$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	if ... then ...	implication
$\neg$	not	negation
$\top$	true	
$\bot$	false	

Syntax of formulæ

Definition. The formulæ of propositional logic, for a given set $\{p, q, r, \ldots\}$ of **propositional atoms**, is given as follows:

- every **propositional atom** a formula, and so are $\top$ and $\bot$;
- if A and B are formulæ, then so are $(A \wedge B)$ and $(A \vee B)$ and $(A \rightarrow B)$;
- if A is a formula, then so is $(\neg A)$;

(We shall often omit brackets if the meaning is clear.)

Meta-variables and object-variables

- The greek letters $A, B, \ldots$ are **meta-variables**: they are **not** formulæ—they part of our mathematician's English.
- By contrast, the propositional atoms $p, q, \ldots$ are **object-variables**: they **are** formulæ.

Meta-language and object-language

Consider the following sentence:

> The Java program P runs faster than the Java program Q, because P has a better handling of the variable `counter`.

- Java is the object-language, i.e. the language **about** which we speak. `counter` is an object-variable, because it belongs to Java.
- IT-English is the meta-language, i.e. the language **in** which we speak. P and Q are meta-variables.

Meta-language and object-language

Back to logics:

- Mathematician’s (or logician’s) English (or German or. . .) is our meta-language. $A, B, \ldots$ are meta-variables.
- Formulæ and similar things form the object-language. $p, q, \ldots$ are object-variables.

Semantics

- So far, we have discussed the **syntax**, i.e. the rules defining the language (of formulæ). But what is the meaning of a formula?
- Semantics is the study of meanings.

Semantics

- For example, in computability theory, the meaning of a program (or Turing machine or abacus machine...) is a function from the natural numbers to the natural numbers.
- English sentences also have a meaning (but it is extremely hard to capture mathematically).

Semantics of logical formulæ

- In logics, meaning is often described by a **satisfaction relation**

$$M \models A$$

  that describes when a **situation** M satisfies a formula A.
- It varies between logics what formulæ and situations are.

Situations

Definition. A **situation** M in propositional logic (also called "valuation") assigns to each propositional atom p a value $[\![p]\!]_M \in \{0, 1\}$.

The satisfaction relation

Definition. The **satisfaction relation** $\models$ is defined as follows:

$$M \models A \wedge B \text{ iff } M \models A \text{ and } M \models B$$

$$M \models A \vee B \text{ iff } M \models A \text{ or } M \models B$$

$$M \models A \rightarrow B \text{ iff } \text{ whenever } M \models A \text{ then } M \models B$$

$$M \models \neg A \text{ iff } M \not\models A$$

$$M \models \top \text{ always}$$

$$M \models \bot \text{ never}$$

$$M \models p \text{ iff } [\![p]\!]_M = 1$$

The satisfaction relation

Definition. A situation M is said to **satisfy** a formula A if $M \models A$.

Definition. A situation M is said to satisfy a set Γ of formulæ if M satisfies every formula in Γ. In this case, we write

$$M \models \Gamma.$$

Models

Definition.

- A situation M that satisfies a formula A is called a **model** of A.
- A situation M that satisfies a set of formulæ Γ is called a **model** of Γ.

Examples

Let M be a situation such that

$$M \models p \qquad M \not\models q \qquad M \models r.$$

Which of the following entailments hold?

1. $M \models p \wedge \neg q$
2. $M \models q \vee \neg r$
3. $M \models p \rightarrow q$
4. $M \models q \rightarrow q$

Truth-table semantics

The satisfaction relation we have just seen can also be presented by using truth-tables:

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

A	$\neg A$
0	1
1	0

Exercise: formalize this.

Validity

Definition. A propositional formula A is called **valid** (or a **tautology**) if it holds in every situation, i.e.

$$M \models A \quad \text{for all situations } M.$$

Example. Which of the formulæ below are valid?

1. $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$
2. $p \vee q \vee \neg r$

Exercise

Show that the formulæ below are tautologies (where $\Box \in \{\wedge, \vee\}$) and $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$:

$((A \Box B) \Box C) \leftrightarrow (A \Box (B \Box C))$	(associativity)
$(A \Box B) \leftrightarrow (B \Box A)$	(commutativity)
$(A \wedge \top) \leftrightarrow A \qquad (A \vee \bot) \leftrightarrow A$	(neutrality)
$(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$	(linear distributivity)
$(\neg A \wedge A) \rightarrow \bot$	(contradiction)
$\top \rightarrow (A \vee \neg A)$	(excluded middle)
$A \leftrightarrow A \Box A$	(idempotency)
$A \rightarrow \top$	
$\bot \rightarrow A$	(ex falso quodlibet).

Remark: this is an axiomatization of Boolean lattices.

Exercise

Show that the formulæ below are tautologies:

$(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$	(DeMorgan)
$(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$	(DeMorgan)
$\top \leftrightarrow \neg\bot$	(DeMorgan)
$\bot \leftrightarrow \neg\top$	(DeMorgan)
$(A \rightarrow B) \leftrightarrow (\neg A \vee B)$	
$(\neg A \rightarrow \bot) \rightarrow A$	(reductio ad absurdum)
$(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$	(contrapositive)
$((A \rightarrow B) \rightarrow A) \rightarrow A$	(Pierce's law).

Satisfiability

Definition. A set of formulæ Γ is called **satisfiable** if it has a model, i.e.

$$M \models \Gamma \quad \text{for some situation } M.$$

Example: Which of the sets below are satisfiable?

- $\{p, \neg q\}$
- $\{p, \neg p\}$

Semantic entailment

Definition. Let $\Gamma = \{A_1, \ldots, A_n\}$ be a set of formulæ, and B a formula. We say that Γ **semantically entails** B and write

$$\Gamma \models B$$

if every model of $\{A_1, \ldots, A_n\}$ is also a model of B.

Remark: sometimes, “entailment” is called “consequence”.

Warning: $\Gamma \models B$ differs from $M \models B$; these conflicting uses of the symbol $\models$ are traditional.

Example

Which of the following entailments hold?

- $\{p, q, r\} \models q$
- $\{\} \models p \vee \neg p$
- $\{p \rightarrow q\} \models p$
- $\{p \wedge \neg p\} \models q$

Exercise: natural deduction

Prove the following facts about semantic entailment. (These are the rules of **natural deduction**, which we shall study soon. The comma stands for union of sets of formulæ.)

$$\frac{\Gamma \models A \quad \Gamma \models B}{\Gamma \models A \wedge B}\wedge i \qquad \frac{\Gamma \models A \wedge B}{\Gamma \models A}\wedge e \qquad \frac{\Gamma \models A \wedge B}{\Gamma \models B}\wedge e$$

$$\frac{\Gamma, A \models B}{\Gamma \models A \rightarrow B}\rightarrow i \qquad \frac{\Gamma \models A \rightarrow B \quad \Gamma \models A}{\Gamma \models B}\rightarrow e$$

$$\frac{\Gamma \models \bot}{\Gamma \models A}\bot e \qquad \frac{\Gamma, \neg A \models \bot}{\Gamma \models A} RAA$$

Example: modus ponens

We prove

$$\frac{\Gamma \models A \to B \quad \Gamma \models A}{\Gamma \models B} \to e$$

This is the famous **modus ponens** already known to the ancient Greeks.

Proof: Suppose that $M \models \Gamma$. Because of the two assumptions, we have $M \models A \to B$ and $M \models A$. By definition, the statement $M \models A \to B$ means that $M \models B$ whenever $M \models A$. So $M \models B$.

Multiple conclusions

Definition. Let $\Gamma = \{A_1, \ldots, A_n\}$ and $\Delta = \{B_1, \ldots, B_m\}$ be sets of formulæ. We say that Γ **semantically entails** Δ and write

$$\Gamma \models \Delta$$

if every model of $A_1, \ldots, A_n$ satisfies **at least one** B_i in Δ.
Note that this is the same as saying that

$$A_1 \wedge \ldots \wedge A_n \models B_1 \vee \ldots \vee B_m.$$

Examples

Which of the following entailments hold?

- $\{p \vee q\} \models \{p, q\}$
- $\{\} \models \{p, q \rightarrow p\}$
- $\{p, \neg p\} \models \{\}$
- $\{\} \models \{p, \neg p\}$

Example: right weakening

Claim: whenever $\Gamma \models \Delta$ and $\Delta \subseteq \Delta'$, it holds that $\Gamma \models \Delta'$. Short notation:

$$\frac{\Gamma \models \Delta}{\Gamma \models \Delta'} \text{ if } \Delta \subseteq \Delta'.$$

Is the claim true?

Exercise: sequent calculus

Prove the following. (These are rules of the **sequent calculus**, which we shall study later in this course.)

$$\frac{}{A \models A}\,Ax \qquad \frac{\Gamma_2 \models \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \models \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_3 \models \Delta_1, \Delta_2, \Delta_3}\,Cut$$

$$\frac{\Gamma, A, B \models \Delta}{\Gamma, A \wedge B \models \Delta}\,L\wedge \qquad \frac{\Gamma \models A, \Delta \quad \Gamma' \models B, \Delta'}{\Gamma, \Gamma', \models A \wedge B, \Delta, \Delta'}\,R\wedge$$

$$\frac{\Gamma, A \models \Delta \quad \Gamma', B \models \Delta'}{\Gamma, \Gamma', A \vee B \models \Delta, \Delta'}\,L\vee \qquad \frac{\Gamma \models A, B, \Delta}{\Gamma \models A \vee B, \Delta}\,R\vee$$

$$\frac{\Gamma \models A, \Delta \quad \Gamma', B \models \Delta'}{\Gamma, \Gamma', A \to B \models \Delta, \Delta'}\,L \to \qquad \frac{\Gamma, A \models \Delta, B}{\Gamma \models A \to B, \Delta}\,R \to$$

Example: the cut rule

The famous cut rule, which we shall study in depth later, states that whenever

$$\Gamma_2 \models \Delta_1, A, \Delta_3 \quad \text{and} \quad \Gamma_1, A, \Gamma_3 \models \Delta_2,$$

then

$$\Gamma_1, \Gamma_2, \Gamma_3 \models \Delta_1, \Delta_2, \Delta_3.$$

Short notation:

$$\frac{\Gamma_2 \models \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \models \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_3 \models \Delta_1, \Delta_2, \Delta_3}.$$

Validity of the cut rule

Suppose that $\Gamma_2 \models \Delta_1, A, \Delta_3$ and $\Gamma_1, A, \Gamma_3 \models \Delta_2$. To see that $\Gamma_1, \Gamma_2, \Gamma_3 \models \Delta_1, \Delta_2, \Delta_3$, assume that $M \models \Gamma_1, \Gamma_2, \Gamma_3$. Because $\Gamma \vdash \Gamma_2$, the situation M satisfies at least one formula in Δ_1, A, Δ_3.

- Case 1: $M \models A$. In this case, we have $M \models \Gamma_1, A, \Gamma_3$, and therefore $M \models \Delta_2$. By right weakening $M \models \Delta_1, \Delta_2, \Delta_3$.
- Case 2: $M \models \Delta_1, \Delta_3$. In this case, the claim follows directly from right weakening.

Entailment, validity, and satisfiability

The semantic entailment relation $\models$ is convenient for expressing validity and unsatisfiability. Before we explain this, we introduce two abbreviations: we write

$$\models \Delta$$

instead of $\{\} \models \Delta$, and

$$\Gamma \models$$

instead of $\Gamma \models \{\}$.

Entailment, validity, and satisfiability

Observation: we have

- $\models A$ if and only if A is valid, and
- $\Gamma \models$ if and only if Γ is unsatisfiable.