Closing the circle

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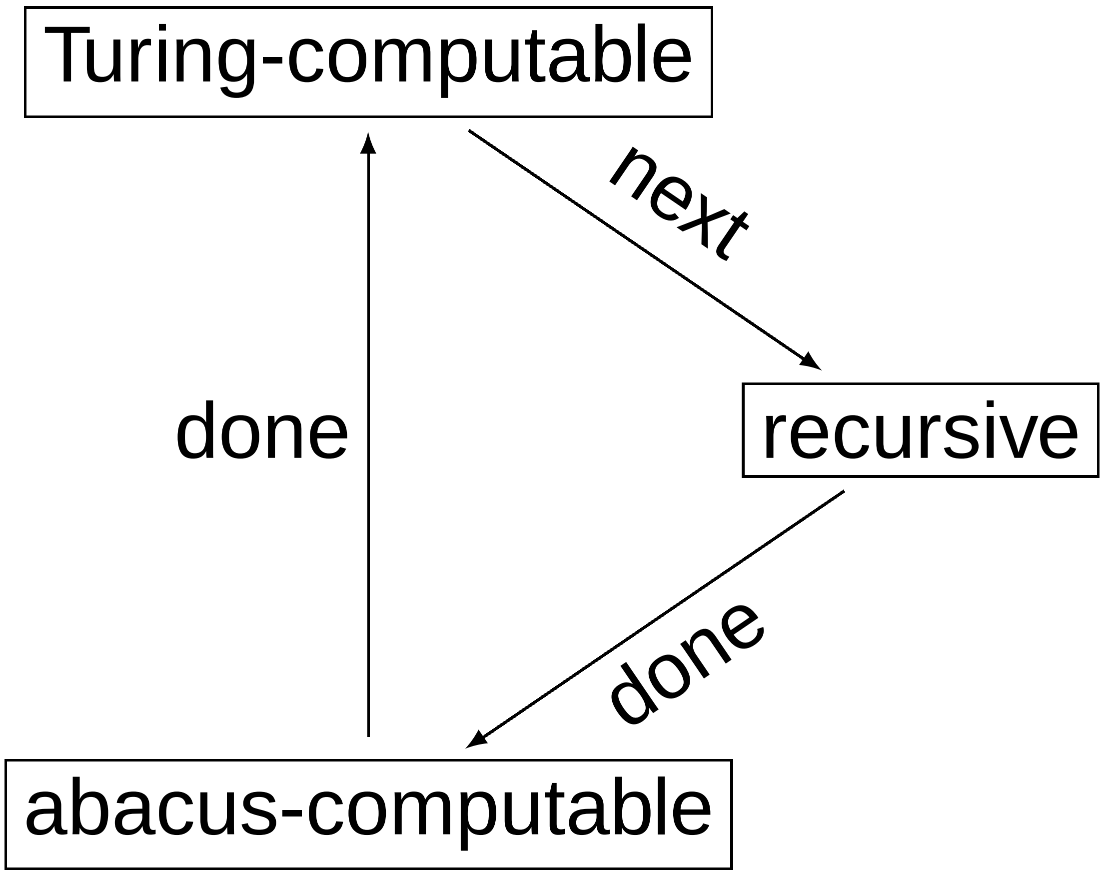

Overview

1. Encode configurations of TM's.
2. Encode TM's themselves.
3. Define a primitive recursive function

   $$conf(m, x, t)$$

   that yields the configuration reached by the TM with code m on input x after time t.
4. Use $conf(m, x, t)$ to define the recursive function computed by the TM.

The Wang encoding for tapes

To encode the tape, we use

- a **left number**, which results from interpreting the tape left of the scanned square as a binary numeral, prefixed by infinitely many superfluous 0's;
- a **right number**, which results from interpreting the rest of the tape, consisting of the scanned square and the portion to its right, as a binary numeral **written backwards**.

Encoding the initial tape

- For the sake of presentation, suppose that the TM takes only one argument, x.
- Then the initial tape has one block of $x+1$ strokes and is otherwise blank, and the leftmost stroke is scanned.
- So the left number is 0, and the right number is

$$2^0 + 2^1 + 2^2 + \cdots + 2^{x-1} + 2^x \;=\; 2^{x+1} \dot{-} 1.$$

- We define a primitive recursive function

$$start(x) = 2^{x+1} \dot{-} 1.$$

Computing the scanned symbol

Let r be the right number. The scanned symbol is

- 0 if the binary representation of r ends with 0, i.e. if r is even.
- 1 if the binary representation of r ends with 1, i.e. if r is odd.
- So the scanned symbol is the remainder of dividing r by 2:

$$scan(r) = rem(r, 2).$$

- As seen earlier, rem is primitive recursive; so the same is true for $scan$.

Writing a 0

Suppose the action is W_0.

- The left number remains the same.
- If the scanned square already contains 0, the right number remains the same; otherwise, it is decreased by 1.
- Letting p be the left number and r the right number, we have

$$newleft_0(p,r) = p$$
$$newright_0(p,r) = r \dot{-} scan(r).$$

Writing a 1

In a similar way, we get a primitive recursive functions for writing a 1:

$$newleft_1(p, r) = p$$

$$newright_1(p, r) = r + 1 \dot{-} scan(r).$$

Moving left: new left number

- Let p be the pre-move left number, and let p^* be the post-move left number.
- The binary representation of p^* is obtained by chopping of the last 0 or 1.
- This means that p^* is p divided by 2 (and rounded down), so p^* is given by

$$newleft_L(p, r) = quo(p, 2).$$

Moving left: new right number

- Let r be the pre-move right number, and let r^* be the post-move right number.
- Let p_0 be the symbol to the left of the scanned square.
- The binary representation of r^* is obtained from the one for r by appending p_0, so

$$r^* = 2r + p_0.$$

- We have $p_0 = rem(p, 2)$; so r^* is given by

$$newright_L(p, r) = 2r + rem(p, 2).$$

Moving right

By reversing the rôles of p and r, we get the functions for moving right:

$$newleft_R(p,r) = 2p + rem(r,2)$$
$$newright_R(p,r) = quo(r,2).$$

Codes for the actions

Before we proceed, we encode the actions as follows:

action	code
W_0	0
W_1	1
L	2
R	3.

The action as an extra argument

We can now define new versions of $newleft$ and $newright$ that take the action as an extra argument:

$$newleft(p, r, a) = \begin{cases} p & \text{if } a = 0 \text{ or } a = 1 \\ quo(p, 2) & \text{if } a = 2 \\ 2p + rem(r, 2) & \text{if } a = 3 \end{cases}$$

This is a **definition by cases**, so $newleft$ is primitive recursive. Similarly for $newright$.

Encoding configurations

- A configuration consists of a tape **and a state**.
- So a configuration can be represented as a triple (p, q, r), where is p and r are left and right numbers, and q is a state.
- We can use the primitive recursive encoding $c = triple(p, q, r) = 2^p \cdot 3^q \cdot 5^r$ and its primitive recursive decodings

$$\begin{aligned} left &= lo(c, 2) \\ state &= lo(c, 3) \\ right &= lo(c, 5). \end{aligned}$$

Extracting the final value

- Suppose that the TM halts in a standard final configuration $c = triple(p, q, r)$.
- If the result is y, then there is a single block with $y + 1$ strokes, which are the binary representation of r; so

$$r \ = \ 2^{y+1} \dot{-} 1.$$

- So $y = lo(r + 1, 2) \dot{-} 1$, i.e. y is given by the primitive recursive function

$$value(c) \ = \ lo(right(c) + 1, 2) \dot{-} 1.$$

Testing for standard final configurations

In a standard final configuration $c = triple(p, q, r)$, we have $p = 0$, and the previous slide implies that

$$\exists y < r . r = 2^{y+1} \dot{-} 1.$$

So c represents a s.f.c. iff the relation

$$is\_std(c) \text{ iff } left(c) = 0 \text{ and } \exists y < right(c) . right(c) = 2^{y+1} \dot{-} 1$$

holds. Because the $\exists$ is bounded, this relation is primitive recursive.

Encoding TM's

We have seen an encoding of TM's before; now we use an improved version. Recall that a TM can be presented by a transition table, e.g.

	0	1
q_1	$W_1 q_1$	$L q_2$
q_2	$W_1 q_2$	$L q_3$
q_3	$W_1 q_3$	

We use the convention that q_1 is the starting state.

Encoding TM's

By introducing a halting state q_0, we can assume that the transition table is defined everywhere. E.g. the table from the previous slide becomes

	0	1
q_0	$W_0 q_0$	$W_1 q_0$
q_1	$W_1 q_1$	$L q_2$
q_2	$W_1 q_2$	$L q_3$
q_3	$W_1 q_3$	$W_1 q_0.$

The table can be written as a list, e.g. $(W_0, q_0, W_1, q_0, W_1, q_1, L, q_2, W_1, q_2, L, q_3, W_1, q_3, W_1, q_0).$

Encoding TM's

The entries of the list can be represented by natural numbers, e.g.

$$(W_0, q_0, W_1, q_0, W_1, q_1, L, q_2, W_1, q_2, L, q_3, W_1, q_3, W_1, q_0)$$

becomes

$$(0, 0, 1, 0, 1, 1, 2, 2, 1, 2, 2, 3, 1, 3, 1, 0).$$

This list can be encoded into a natural number m which is the code of the TM, e.g.

$$2^0 \cdot 3^0 \cdot 5^1 \cdot 7^0 \cdot 11^1 \cdot 13^1 \cdot 17^2 \cdot \cdots .$$

Using the encoding

$$(W_0, q_0, W_1, q_0, W_1, q_1, L, q_2, W_1, q_2, L, q_3, W_1, q_3, W_1, q_0)$$
$$(0, 0, 1, 0, 1, 1, 2, 2, 1, 2, 2, 3, 1, 3, 1, 0).$$

- The action when scanning symbol i in state q is given by entry number $4q + 2i$.
- The next state is given by entry number $4q + 2i + 1$.
- We have primitive recursive functions

$$action(m, q, r) = entry(m, 4q + 2 \cdot scan(r))$$
$$newstate(m, q, r) = entry(m, 4q + 2 \cdot scan(r) + 1).$$

Configuration after t steps

Next, we define a primitive recursive function $conf(m, x, t)$ that returns the configuration reached by TM with code m on input x after t steps.

- After 0 steps we have

$$conf(m, x, 0) = triple(0, 1, start(x)).$$

- We define

$$conf(m, x, t + 1) = newconf(m, conf(m, x, t)).$$

Defining $newconf(m, c)$

1. Apply $left$, $state$, and $right$ to c to obtain the left number p, the number q of the state, and the right number r.
2. Apply $action$ and $newstate$ to (m, q, r) to obtain the number a of the action, and the number q^* of the new state.
3. Let $newconf(m, c) =$ $triple(newleft(p, r, a), q^*, newright(p, r, a))$.

We used only composition, so $newconf$ is primitive recursive.

Halting in standard configuration

- The TM is halted when $state(conf(m, x, t)) = 0$.
- So, letting

$$stdh(m, x, t) = \begin{cases} 0 & \text{if } state(conf(m, x, t)) = 0 \\ & \quad \text{and } is\_std(conf(m, x, t)) \\ 1 & \text{otherwise,} \end{cases}$$

  the machine is halted in a standard configuration iff $stdh(m, x, t) = 0$.
- This is a definition by cases, so the function $stdh$ is primitive recursive.

The time of halting

The time (if any) when the machine halts in a standard configuration is

$$halt(m,x) = \begin{cases} \text{the least } t \text{ such that } stdh(m,x,t) = 0 & \text{if such a } t \text{ exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The function $halt$ is recursive, because it is defined by minimization over a (primitive) recursive function ($stdh$).

Putting it all together

- Let $F(m, x) = value(conf(m, x, halt(m, x)))$.
- $F(m, x)$ is the value of the function computed by the TM with code m for argument x.
- F is recursive, because it is defined by composition from recursive functions.
- Let $f(x) = F(m, x)$.
- f is the function computed by the TM with code m, and f is recursive.

Theorem

So we have proved:

Theorem. Every Turing-computable function is recursive.

This closes the circle.