On the Geometry of Interaction for Classical Logic

(with David Pym)

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EPSRC project

“Semantics of classical proofs”, also involving Hyland, Robinson, and Urban.

The non-determinism of cut-reduction

The proof in the middle (attributed to Lafont) reduces to both Φ_1 and Φ_2:

$$
\begin{array}{c} \Phi_1 \\ \vdots \\ \Gamma \vdash \Delta \end{array}
\quad \succcurlyeq \quad
\dfrac{\dfrac{\begin{array}{c} \Phi_1 \\ \vdots \\ \Gamma \vdash \Delta \end{array}}{\Gamma \vdash A, \Delta}\,\mathsf{WR} \qquad \dfrac{\begin{array}{c} \Phi_2 \\ \vdots \\ \Gamma \vdash \Delta \end{array}}{\Gamma, A \vdash \Delta}\,\mathsf{WL}}{\Gamma \vdash \Delta}\,\mathsf{Cut}
\quad \preccurlyeq \quad
\begin{array}{c} \Phi_2 \\ \vdots \\ \Gamma \vdash \Delta. \end{array}
$$

Therefore, models that preserve meaning along cut reductions are trivial.

Flawed models

- CCC's with a dualizing object, i.e. an object $\perp$ such that the map below has an inverse for every object A.

$$A \longrightarrow ((A \to \perp) \to \perp)$$

  - Problem: such categories are boolean lattices.

- Translations into classical natural deduction.
  - Problem: admit only the left reduction in Lafont's example (call-by-value) or the right one (call-by-name).

Overview

1. Introduction of *order-enriched* models $\mathbf{C}[\![-]\!]$ such that

$$\Phi \preccurlyeq \Psi \implies \mathbf{C}[\![\Phi]\!] \leq \mathbf{C}[\![\Psi]\!].$$

   - Examples: $\mathbf{Rel}_{\otimes}$, $\mathbf{Rel}_{\oplus}$, boolean algebras, proof nets.
   - Soundness & completeness.

2. Main example: extended GoI.

   - Study of weakening and contraction.

Starting point: models of MLL

Symmetric linearly distributive categories for modelling MLL (Cockett & Seely).

- Symmetric monoidal product $\otimes$ for modelling $\wedge$ and left comma.
- Symmetric monoidal product $\oplus$ for modelling $\vee$ and right comma.
- Objects 0 and 1 for modelling $\perp$ and $\top$.
- Optionally, maps as below for modelling $\neg$ (yields $*$-autonomous categories).

$$\neg A \otimes A \longrightarrow 0 \qquad\qquad 1 \longrightarrow A \oplus \neg A.$$

Modelling weakening and contraction

- A type-indexed family of symmetric monoids

$$A \oplus A \xrightarrow{\nabla_A} A \xleftarrow{[]_A} 0$$

  satisfying the evident coherence conditions.

- A type-indexed family of symmetric co-monoids

$$A \otimes A \xleftarrow{\Delta_A} A \xrightarrow{\langle\rangle_A} 1$$

  satisfying the evident coherence conditions.

Example: associativity

The associativity law of the monoids corresponds to

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta_1, (A, A), A, \Delta_2\end{matrix}}{\Gamma \vdash \Delta_1, A, A, \Delta_2}\,\mathsf{CR}}{\Gamma \vdash \Delta_1, A, \Delta_2}\,\mathsf{CR} \quad \equiv \quad \dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta_1, A, (A, A), \Delta_2\end{matrix}}{\Gamma \vdash \Delta_1, A, A, \Delta_2}\,\mathsf{CR}}{\Gamma \vdash \Delta_1, A, \Delta_2}\,\mathsf{CR}.$$

Classical categories

Definition 1. *A* Dummett category *is partial-order enriched symmetric linearly distributive category with symmetric monoids and comonoids such that*

1. *$\otimes$, and $\oplus$ are monotonic in both arguments;*

2. *parametric versions of the laws below hold.*

to model cut reductions involving C	*to model cut reductions involving W*
$f \circ \nabla \ \leq \ \nabla \circ (f \oplus f)$	$f \circ [] \ \leq \ []$
$\Delta \circ f \ \leq \ (f \otimes f) \circ \Delta$	$\langle\rangle \circ f \ \leq \ \langle\rangle$

A classical category *is a Dummett category with* $\neg A \otimes A \longrightarrow 0$ *and* $1 \longrightarrow A \oplus \neg A$.

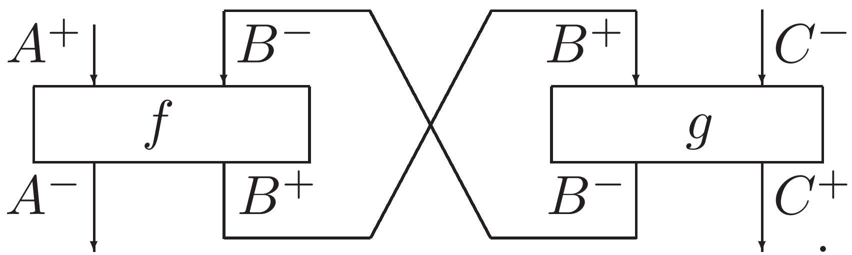
Soundness & completeness

- *Sequent theories*: judgments are of the form below, where Φ and Ψ are proofs of the same sequent.
$$\Phi \preccurlyeq \Psi$$
- $\preccurlyeq$ contains cut-reduction.
- An *interpretation* $\mathbf{C}[\![-]\!]$ is a classical category $\mathbf{C}$ with an object for every atomic formula.
- A *model* is an interpretation $\mathbf{C}[\![-]\!]$ such that $\Phi \preccurlyeq \Psi$ implies $\mathbf{C}[\![\Phi]\!] \leq \mathbf{C}[\![\Psi]\!]$.
- We have soundness and completeness in the evident sense.

The GoI construction

Definition 2. *Given a traced symmetric monoidal category* $\mathbf{C}$, *the category* $\mathrm{GoI}(\mathbf{C})$ *is defined as follows:*

- *Objects are pairs* (A^+, A^-) *of objects of* $\mathbf{C}$*;*
- *A morphism* $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ *of* $\mathrm{GoI}(\mathbf{C})$ *is a morphism* $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ *of* $\mathbf{C}$*;*
- *Composition is defined by* symmetric feedback*; informally,*



The GoI category $\mathrm{GoI}(\mathbf{C})$

Theorem 1. [Joyal/Street/Verity] $\mathrm{GoI}(\mathbf{C})$ *is a compact closed category (*$=$ *a symmetric linearly distributive category with* $\neg$ *such that* $\otimes = \oplus$ *and* $0 = 1$*).*

The extended GoI construction

Theorem 2. [Hasegawa] *If* $\mathbf{C}$ *is a traced compact Dummett category, then* $\mathrm{GoI}(\mathbf{C})$ *is a classical category.*

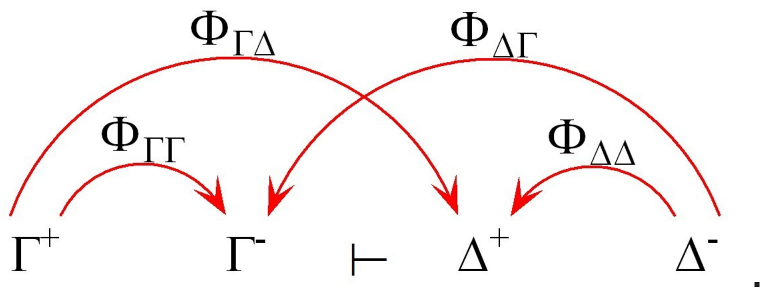
(Generalized version of theorem in our LICS paper.)

Spelling out the extended GoI for $\mathbf{Rel}_{\oplus}$

- Let Φ be a proof of $\Gamma \vdash \Delta$.

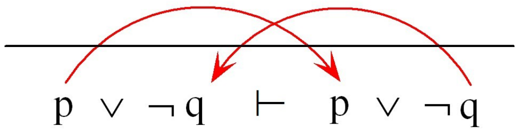
- Let Γ^+ resp. Γ^- be the set of positive resp. negative occurrences of atomic formulæ in Γ. Same for Δ.

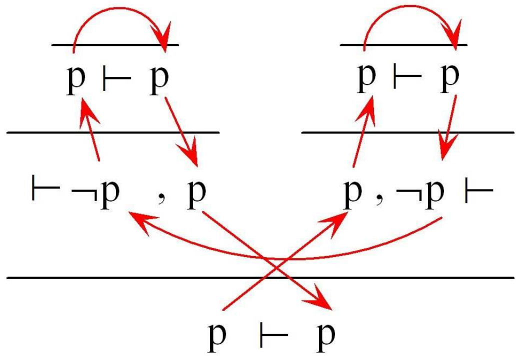
- The denotation of Φ is a quadruple of relations



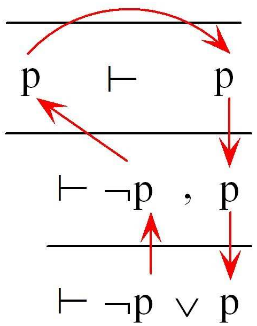
- The order $\leq$ of the classical category is component-wise $\supseteq$.

Finding denotations: examples





EM =



Weakening in $\mathrm{GoI}(\mathbf{Rel}_{\oplus})$

The law
$f \circ []_A \leq []_B$
corresponds to

$$\dfrac{\dfrac{\dfrac{}{\bot \vdash}}{\bot \vdash A}\,\mathsf{WR} \qquad \begin{array}{c}\Phi \\ \vdots \\ A \vdash B\end{array}}{\bot \vdash B}\,\mathsf{Cut} \;\preccurlyeq\; \dfrac{\dfrac{}{\bot \vdash}}{\bot \vdash B}\,\mathsf{WR}$$

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Special
case:

$$\dfrac{\dfrac{\overline{\bot \vdash}}{\bot \vdash q} \qquad \dfrac{\begin{array}{c}\mathrm{EM} \\ \vdots \\ \vdash \neg p \vee p\end{array}}{q \vdash \neg p \vee p}}{\bot \vdash \neg p \vee p} \quad \preccurlyeq \quad \dfrac{\overline{\bot \vdash}}{\bot \vdash \neg p \vee p}$$

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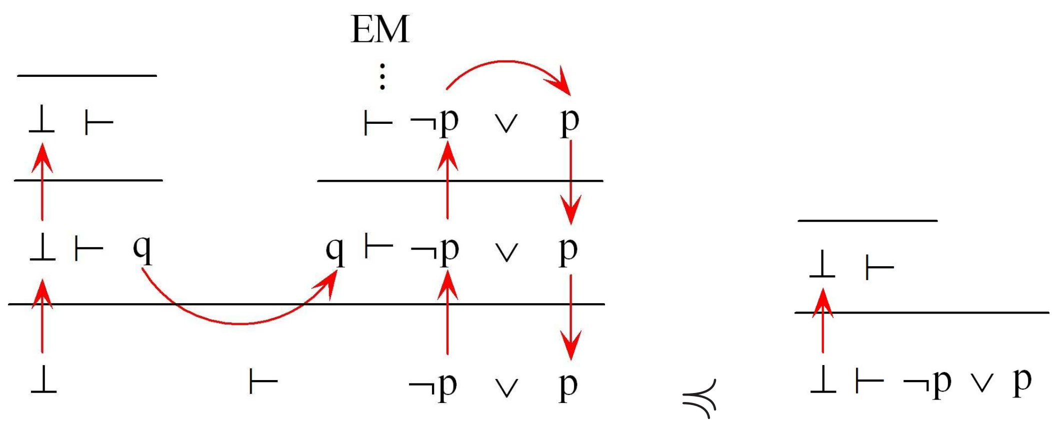

Contraction in $\mathrm{GoI}(\mathbf{Rel}_{\oplus})$

The law
$\Delta_B \circ f \leq (f \otimes f) \circ \Delta_A$
corresponds to

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ A \vdash B\end{array} \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{B \vdash B}}{B, B \vdash B \wedge B}}{B \vdash B \wedge B}\,\text{CL}}{A \vdash B \wedge B}\,\text{Cut} \quad \preccurlyeq \quad \dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ A \vdash B\end{array} \qquad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ A \vdash B\end{array} \quad \dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{B \vdash B}}{B, B \vdash B \wedge B}}{B \vdash B \wedge B}\,\text{Cut}}{A, A \vdash B \wedge B}\,\text{Cut}}{A \vdash B \wedge B}\,\text{CL}$$

:

Contraction in $\mathrm{GoI}(\mathbf{Rel}_{\oplus})$

The law $\Delta_B \circ f \leq (f \otimes f) \circ \Delta_A$ corresponds to

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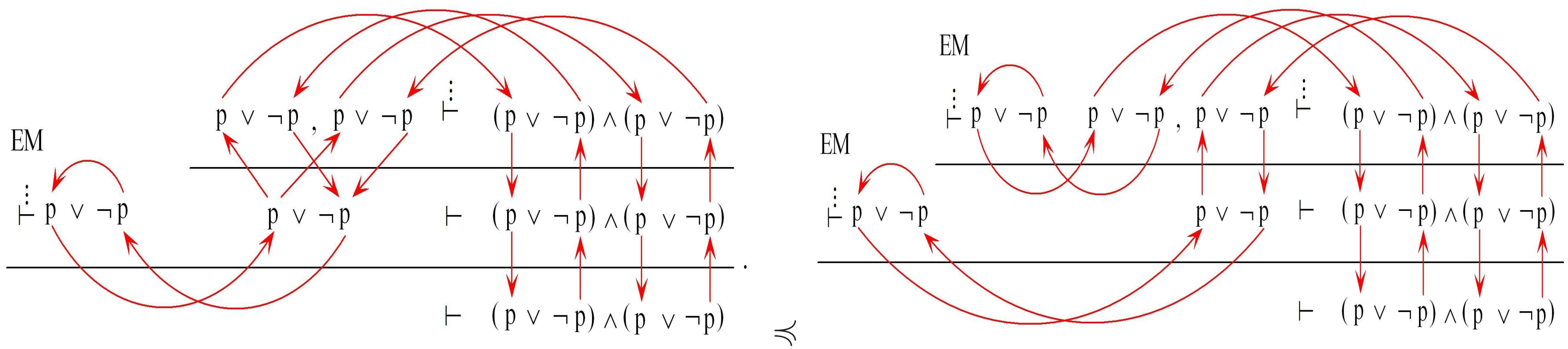
Special case:



Directions

- More non-compact models. Games? (Pym/Ritter.)
- Extension to predicate logic (McKinley)