Intuitionistic logic (Part 2/2)

Summary so far

- Classical proofs of $A \vee B$ or $\exists x.A$ fail to produce witnesses.
- Solution to this problem: get rid of RAA/LEM.
- The result is called **intuitionistic logic**.
- IL has a ND system which looks like the one for classical logic minus RAA.
- IL has a Kripke semantics.

Monotonicity of forcing

Proposition. In any Kripke model of IL, if x and y are worlds such that

$$x \leq y,$$

then, for every formula A,

$$x \Vdash A \text{ implies } y \Vdash A.$$

Proof. (Sketch.) By induction on A; the case $A = p$ follows from the monotonicity of L; the case $A = B \to C$ is interesting, because it relies on the transitivity of $\leq$; the other cases are straightforward.

Soundness and completeness

To avoid confusion, we write

- $\Gamma \vdash_I A$ and $\Gamma \models_I A$ for syntactic and semantic entailment in intuitionistic logic.
- $\Gamma \vdash_C A$ and $\Gamma \models_C A$ for syntactic and semantic entailment in classical logic.

Proposition.[Soundness] $\Gamma \vdash_I A$ implies $\Gamma \models_I A$.

Theorem.[Completeness] $\Gamma \models_I A$ implies $\Gamma \vdash_I A$.

Proof of soundness

By induction on the size of the ND proof.

- The only interesting cases are $\rightarrow i$ and $\rightarrow e$.
- The $\rightarrow i$ case relies on the monotonicity of forcing (which in turn relies the transitivity of the accessibility relation).
- The $\rightarrow e$ case uses the reflexivity of the accessibility relation.
- So intuitionistic implication is the reason why the accessibility relation has to be a preorder.

Exercises

Show that the following are provable in intuitionistic logic.

1. $A \vdash \neg\neg A$
2. $\neg\neg\neg A \vdash \neg A$
3. $\neg\neg(A \wedge B) \vdash (\neg\neg A \wedge \neg\neg B)$
4. $\neg\neg(A \to B) \vdash (\neg\neg A \to \neg\neg B)$
5. $\neg\neg\bot \vdash \bot$
6. $\neg\neg\forall x.B \vdash \forall x.\neg\neg B$

(We shall use these later.)

Disjunction and existence property

Proposition.

1. Intuitionistic logic has the disjunction property, i.e., $\vdash_I A \vee B$ implies $\vdash_I A$ or $\vdash_I B$.
2. Intuitionistic predicate logic has the existence property, i.e., $\vdash_I \exists x.A$ implies that $\vdash_I A[t/x]$ for some t.

The rôles of $\vee$ and $\exists$

- Because of the disjunction property, an intuitionistic proof of $A \vee B$ requires a choice as to whether we prove A or B.
- Similarly, a proof of $\exists x.A$ requires a witness t such that $A[t/x]$ is true.
- So, intuitively, formulæ of the form $A \vee B$ or $\exists x.A$ carry most of the burden of constructiveness.
- The next theorem that makes this intuition precise.

Negative formulæ

Definition. A formula A is called **negative** if it contains no $\vee$, no $\exists$, and if occurrences of atomic formulæ (but not $\bot$) are negated.

- Examples: $\neg\neg\neg p \to \neg p$, $\neg p \wedge \bot$.
- Non-examples: $\neg\neg p \to p$.

IL on negative formulæ

Theorem. If A is a negative formula, then $\neg\neg A \vdash_I A$.

Intuitively, proof by contradiction works even in IL, if the formula involved contains neither $\vee$ nor $\exists$, and all atoms are negated.

Proof. By induction on A, using the facts from the last exercise.

Gödel’s translation

Remarkably, there is a translation, taking every formula A to a formula A°, that allows to describe classical provability in terms of intuitionistic provability, i.e.

$$\Gamma \vdash_C A \quad \text{iff} \quad \Gamma^\circ \vdash_I A^\circ$$

where Γ° means that the translation is applied to every formula in Γ.

Gödel's translation

Definition. Gödel's translation (also Gentzen) is the map from formulæ to formulæ defined by the following rules:

$$\bot^\circ = \bot$$

$$p^\circ = \neg\neg p \qquad \text{for atomic } p$$

$$(A \wedge B)^\circ = A^\circ \wedge B^\circ$$

$$(A \vee B)^\circ = \neg(\neg(A^\circ) \wedge \neg(B^\circ))$$

$$(A \to B)^\circ = A^\circ \to B^\circ$$

$$(\forall x.A)^\circ = \forall x.(A^\circ)$$

$$(\exists x.A)^\circ = \neg\forall x.\neg(A^\circ)$$

Gödel's translation

The point about Gödel's translation is that it removes $\vee$ and $\exists$, and makes sure that all atomic formulæ occur under a negation:

Proposition. For every formula A, the formula A° is negative.

Gödel's translation

Theorem.

$$\Gamma \vdash_C A \text{ iff } \Gamma^\circ \vdash_I A^\circ.$$

That is, classical logic can be **embedded** into intuitionistic logic.

Proof.

First, we prove the $\Leftarrow$ direction. If $\Gamma^\circ \vdash_I A^\circ$, then $\Gamma^\circ \vdash_C A^\circ$, because intuitionistic ND is a subsystem of classical ND. It is easy to see that A and A° have the same classical truth-value, so $\Gamma \vdash_C A$.

The $\Rightarrow$ direction is proved by induction on the derivation of $\Gamma \vdash_C A$, making crucial use of the earlier theorem which states that $\neg\neg B \vdash_I B$ for negative formulæ B.

Intuitionistic sequent calculus

The classical sequent calculus allows to prove the law of the excluded middle:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,Ax}{\vdash \neg A, A}\,R\neg}{\vdash \neg A \vee A}\,R\vee$$

Note the use of **multiple conclusions**.

Fact (without proof): the **single-conclusioned** sequent calculus on the next slide is sound and complete for IL. (It is the previously-seen minimal sequent calculus plus EFQ.)

An intuitionistic sequent calculus

$$\frac{}{\Gamma, A \vdash A}\,Ax \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,EFQ$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,L\wedge \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,R\wedge$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\,L\vee \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\,(i = 1, 2) R\vee$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\,L \to \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,R \to$$

$$\frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x.A \vdash B}\,L\forall \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}\,R\forall$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \exists x.A \vdash B}\,L\exists \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x.A}\,R\exists$$