Intuitionistic logic
Motivation for intuitionistic logic

- As hinted earlier, proof by contradiction (RAA) is contentious.

- As shown before, (RAA) is interderivable with the law of the excluded middle

\[ A \lor \neg A \quad LEM. \]

- We shall now see an example why \( LEM \) (and therefore \( RAA \)) is contentious.
Motivation for intuitionistic logic

**Proposition.** There exist two irrational numbers $a, b$ such that $a^b$ is rational.
Constructivism

- The proof we have seen is deemed to be not constructive.

- An attack on the law of the excluded middle was launched by the famous mathematician-logician L.E.J. Brouwer in the early 1900’s.

- Brouwer’s mathematics and logics are called intuitionistic.

In this context, the traditional non-constructive mathematics and logics are called classical.
The idea in constructive logic is that we can only consider a statement to be true if we have a proof for it.

This idea is made precise by Heyting’s interpretation of proofs:
Heyting interpretation

- A proof of $A \land B$ is a pair $(\Phi, \Psi)$ where $\Phi$ is a proof of $A$ and $\Psi$ is a proof of $B$.

- A proof of $A \lor B$ is a proof of $A$ or a proof of $B$.

- A proof of $A \rightarrow B$ is a method for turning a proof of $A$ into a proof of $B$.

- A proof of $\forall x . A$ is a method for turning any witness $t$, into a proof of $A[t/x]$.

- A proof of $\exists x . A$ consists of a witness $t$ and a proof $\Phi$ of $A[t/x]$. 
ND and Heyting interpretation

The gist of the Heyting interpretation is captured by the natural deduction rules \textit{minus} RAA:
Given a proof $\Phi$ of $A$ and a proof $\Psi$ of $B$, we have a proof of $A \land B$.

Given a proof $\Phi$ of $A \land B$, we have a proof of $A$ and a proof of $B$.

So, to have a proof of $A \land B$ is to have a proof of $A$ and a proof of $B$. 
ND and Heyting interpretation: →

\[
\begin{align*}
&\frac{[A]}{\vdash \Phi} \\
&\frac{B}{A \rightarrow B \rightarrow i}
\end{align*}
\]

Given a method \( \Phi \) for turning a proof of \( A \) into a proof of \( B \), we have a proof of \( A \rightarrow B \).

\[
\begin{align*}
&\vdash \Phi \\
&\vdash \Psi \\
&\vdash A \rightarrow B \\
&\vdash A \\
&\frac{A \rightarrow B \rightarrow e}{B}
\end{align*}
\]

Given a proof \( \Phi \) of \( A \rightarrow B \), we have a method for turning any proof \( \Psi \) of \( A \) into a proof of \( B \).

So, to have a proof of \( A \rightarrow B \) is to have a method for turning any proof of \( A \) into a proof of \( B \).
Given a proof of $A$ for an arbitrary $x$ (i.e., a method for proving $A[t/x]$ for any $t$), we have a proof of $\forall x. A$.

Given a proof of $\forall x. A$, we have a method for proving of $A[t/x]$ for any $t$.

(Warning: the side conditions are omitted in the above presentation of the rules.) So, to have a proof of $\forall x. A$ is to have a method for proving $A[t/x]$ for any $t$. 
ND and Heyting interpretation: \( \lor \)

Given a proof of \( \Phi \) of \( A \) (or of \( B \)), we have a proof of \( A \lor B \).

Given a proof \( \Phi \) of \( A \lor B \) and methods \( \Psi_1 \) resp. \( \Psi_2 \) for turning proofs of \( A \) resp. \( B \) into proofs of \( C \), we have a proof of \( C \).
The disjunction property

- Introduction and elimination rules for \( \vee \) do not imply the disjunction property, which states that

\[
\text{if } \vdash A \lor B, \text{ then } \vdash A \text{ or } \vdash B.
\]

- To see this, note that in classical propositional logic, we have neither \( \vdash p \) nor \( \vdash \neg p \) for an atomic formula \( p \).

- But the disjunction property holds for intuitionistic logic, as we shall see later.
ND and Heyting interpretation: ∃

Given a proof $\Phi$ of $A[t/x]$ for some witness $t$, we have a proof of $\exists x . A$.

Given a proof $\Phi$ of $A[t/x]$, and a method for turning a proof of $A$ (for arbitrary $x$) into a proof of $B$, we get a proof of $B$.

(Warning: the side conditions are omitted in the above presentation of the rules.)
The existence property

- Introduction and elimination rules for \( \exists \) do not imply the **existence property**, which states that

  \[
  \text{if } \vdash \exists x.A, \text{ then } \vdash A[t/x] \text{ for some } t.
  \]

- But the existence property holds for intuitionistic logic.
Ex falso quodlibet

The elimination rule for $\bot$ is contentious, but not as contentious as $RAA$. (As seen earlier, $RAA$ implies $\bot e$; the converse is false, as we shall see.)

$\phi$

$\bot e$

If $\phi$ is a proof of a contradiction, we are allowed to turn this into a proof of any formula $A$.

This rule is allowed in intuitionistic logic, but not in minimal logic.
Summary of ND for IL

For simplicity, we shall focus on propositional logic.

\[
\begin{aligned}
A & \quad B \\
\hline
A \land B
\end{aligned}
\quad \quad
\begin{aligned}
A \land B \\
\hline
A
\end{aligned}
\quad \quad
\begin{aligned}
A \land B \\
\hline
B
\end{aligned}
\]

\[
\begin{aligned}
A \\
\hline
A \lor B
\end{aligned}
\quad \quad
\begin{aligned}
B \\
\hline
A \lor B
\end{aligned}
\quad \quad
\begin{aligned}
A \lor B \\
\hline
C
\end{aligned}
\]

\[
\begin{aligned}
[A] \\
\vdots \\
B \\
\hline
A \rightarrow B
\end{aligned}
\quad \quad
\begin{aligned}
[B] \\
\vdots \\
\vdots \\
\hline
B
\end{aligned}
\quad \quad
\begin{aligned}
A \rightarrow B \\
A \\
\hline
B
\end{aligned}
\]

\[
\begin{aligned}
\vdash \quad \vdash e
\end{aligned}
\]

\[
\begin{aligned}
A \lor B \\
\hline
C
\end{aligned}
\quad \quad
\begin{aligned}
A \lor B \\
\hline
C
\end{aligned}
\quad \quad
\begin{aligned}
A \rightarrow B \\
A \\
\hline
B
\end{aligned}
\]

\[
\begin{aligned}
\vdash \quad \vdash e
\end{aligned}
\]

\[
\begin{aligned}
A \rightarrow B \\
A \\
\hline
B
\end{aligned}
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\begin{aligned}
\vdash \quad \vdash e
\end{aligned}
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\[
\begin{aligned}
A \rightarrow B \\
A \\
\hline
B
\end{aligned}
\]

\[
\begin{aligned}
\vdash \quad \vdash e
\end{aligned}
\]

\[
\begin{aligned}
A \rightarrow B \\
A \\
\hline
B
\end{aligned}
\]

\[
\begin{aligned}
\vdash \quad \vdash e
\end{aligned}
\]
Alternative version

\[
\begin{align*}
\Gamma 
&\vdash A \\
\Gamma 
&\vdash B
\end{align*}
\quad \begin{array}{c}
\wedge i
\end{array}
\begin{align*}
\Gamma 
&\vdash A \\
\Gamma 
&\vdash A \land B
\end{align*}
\quad \begin{array}{c}
\wedge e
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\begin{align*}
\Gamma 
&\vdash A \land B \\
\Gamma 
&\vdash A
\end{align*}
\quad \begin{array}{c}
\wedge e
\end{array}
\begin{align*}
\Gamma 
&\vdash A \land B \\
\Gamma 
&\vdash B
\end{align*}
\quad \begin{array}{c}
\wedge e
\end{array}
\begin{align*}
\Gamma 
&\vdash A \\
\Gamma 
&\vdash A \lor B
\end{align*}
\quad \begin{array}{c}
\lor i
\end{array}
\begin{align*}
\Gamma 
&\vdash B \\
\Gamma 
&\vdash A \lor B
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\quad \begin{array}{c}
\lor i
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\begin{align*}
\Gamma 
&\vdash A \lor B \\
\Gamma 
&\vdash A
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\quad \begin{array}{c}
\lor i
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\begin{align*}
\Gamma 
&\vdash A \lor B \\
\Gamma 
&\vdash B
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\quad \begin{array}{c}
\lor i
\end{array}
\begin{align*}
\Gamma 
&\vdash A \lor B \\
\Gamma 
&\vdash B
\end{align*}
\quad \begin{array}{c}
\lor i
\end{array}
\begin{align*}
\Gamma 
&\vdash \bot
\Gamma 
&\vdash A
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\quad \begin{array}{c}
\bot e
\end{array}
\begin{align*}
\Gamma 
&\vdash A \lor B \\
\Gamma 
&\vdash C \\
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&\vdash C
\end{align*}
\quad \begin{array}{c}
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\begin{align*}
\Gamma 
&\vdash A \lor B \\
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&\vdash C
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\quad \begin{array}{c}
\lor e
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\begin{align*}
\Gamma 
&\vdash A \\
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\quad \begin{array}{c}
\rightarrow i
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\begin{align*}
\Gamma 
&\vdash A \rightarrow B
\Gamma 
&\vdash A
\end{align*}
\quad \begin{array}{c}
\rightarrow e
\end{array}
\begin{align*}
\Gamma 
&\vdash A \rightarrow B
\Gamma 
&\vdash B
\end{align*}
Semantics of IL?

- \( \Gamma \vdash A \) is provable in ND for **classical** propositional logic iff \( \Gamma \models A \) in the sense of the truth-table semantics.

- The absence of \( RAA \) from IL suggests that IL proves fewer judgments \( \Gamma \vdash A \), and is therefore incomplete w.r.t. the truth-table semantics.

- Is there a semantics w.r.t. which IL is complete?
Remarkably, a variation of Kripke models for modal logic also works for IL. Three changes are enough:

1. The accessibility relation $R$ is a preorder, i.e. reflexive and transitive. We shall write $\leq$ instead of $R$.

2. The labelling function is required to be monotonic, i.e. $L(x) \subseteq L(y)$ whenever $x \leq y$.

3. We shall need to change the forcing semantics of implication.
Heuristic motivation

- An idealized mathematician (traditionally called the “creative subject”) explores the possible worlds.

- The preorder can be seen to describe (branching) time: $x < y$ means that world $y$ is later than world $x$.

- The mathematician can only move forward in time; along the way, she discovers true facts.

- If she knows a fact to be true at world $x$, she also knows it to be true in any later world. (That explains why the labelling function is monotonic.)
Kripke models for IL

**Definition.** A (Kripke) model of propositional IL consists of

1. a set $W$, whose elements are called worlds;
2. a preorder $\leq$ on $W$;
3. a monotonic labelling function $L : W \rightarrow P(\text{Atoms})$. 
The semantics of $\land$, $\lor$, $\bot$, and of atomic formulæ, is the same as in basic modal logic:

\[
\begin{align*}
x \models A \land B & \iff x \vdash A \text{ and } x \vdash B \\
x \models A \lor B & \iff x \vdash A \text{ or } x \vdash B \\
x \not\models \bot & \\
x \models p & \iff p \in L(x)
\end{align*}
\]
Semantics of $\rightarrow$

- One can know $A \rightarrow B$ to be true without knowing whether $A$ or $B$ are true.

- However, it does not suffice to look only at the present world: one must know that no later discovery can make $A \rightarrow B$ false.

This motivates the following semantics of $\rightarrow$:

$$x \models A \rightarrow B \text{ iff } \text{ for all } y \text{ with } x \leq y, \text{ if } y \models A \text{ then } y \models B.$$
Semantics of $\rightarrow$:

Let $x$ be a world, and let $p$ and $q$ be atomic formulæ.

1. If $q$ is true at $x$, then $x \models p \rightarrow q$.

2. If $p$ is true and $q$ is false at $x$, then $x \not\models p \rightarrow q$.

3. If both $p$ and $q$ are false at $x$, we must look into the future.
Semantics of $\neg$

As before, we define

$$\neg A = (A \rightarrow \bot).$$

Thus

$$x \models \neg A \quad \text{iff} \quad \text{for all } y \text{ with } x \leq y \text{ we have } y \not\models A.$$

That is, we know $\neg A$ if $A$ never becomes true.
Double negation

**Lemma.** In every Kripke model for IL, it holds for every world \( x \) that

\[
x \models \neg\neg A
\]

if and only if

for all \( y \geq x \) there is a \( z \geq y \) such that \( z \models A \).

**Proof.** See lecture.
Some non-valid formulæ

The following formulæ, which are valid in classical logic, are not valid in IL:

1. $\neg\neg p \rightarrow p$
2. $p \lor \neg p$
3. $\neg(p \land q) \rightarrow (\neg p \lor \neg q)$
4. $(p \rightarrow q) \rightarrow (\neg p \lor q)$. 