Natural deduction for modal logic

Towards ND for modal logic

- Because $\Diamond A \equiv \neg\Box\neg A$, we drop $\Diamond$ for simplicity.
- Fact: if we add the rule below to ND for propositional logic, we get an inference system for basic modal logic.

$$\frac{\Gamma \vdash B}{\Box\Gamma \vdash \Box B}$$

(If $\Gamma = A_1, \ldots A_n$, then $\Box\Gamma$ stands for $\Box A_1, \ldots, \Box A_n$.)

Soundness and completeness

- Soundness: see lecture.
- Completeness: beyond the scope of this lecture.

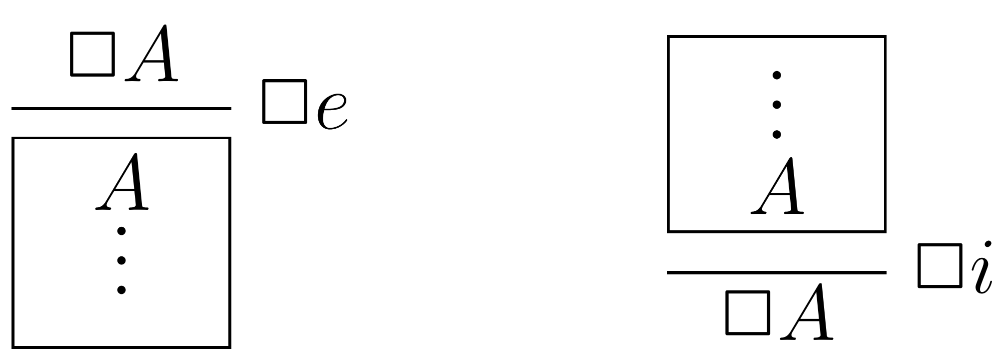
Not yet ND

Issue:

- The rule we just added is not an ND-introduction rule (because $\Box$ is also introduced on the left);
- This is not in the spirit of ND.
- This can be fixed, but it requires boxes (similar the ones used for $\rightarrow i$, but with some important differences).

Introduction and elimination of $\Box$

- Proofs can contain boxes that deal with $\Box$.
- $\Box$-elimination: if we have proved $\Box A$, then we can put A as a **hypothesis** into a box.
- $\Box$-introduction: if we have a box whose **conclusion** is A, we can conclude $\Box A$ outside of the box.



Examples and non-examples

The judgment

$$\Box A \wedge \Box B \vdash \Box(A \wedge B)$$

has an ND proof, but the two judgments below don't (see lecture).

$$\Box A \vdash \Box\Box A$$

$$\Box A \vdash A$$

Caution

The boxes for $\rightarrow$ and $\Box$ differ crucially.

- When we prove $A \rightarrow B$ by using $\rightarrow i$, we can use the hypothesis A inside the box:

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B} \rightarrow i.$$

- Moreover, **any** hypotheses from outside the box can be used inside the box. Example: ND proof of $\vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$ (see lecture).

Caution

- By contrast, when we prove $\Box B$ by using $\Box i$, then the hypotheses in the box must be put inside the box by $\Box e$.
- So we must make sure that we do not confuse the two types of boxes. (E.g. by using different colors, or dashed vs. solid lines.)
- Example: a proof of $\Box(A \rightarrow B) \vdash (\Box A \rightarrow \Box B)$.