

# **On the Geometry of Interaction for Classical Logic**

Carsten Führmann  
(joint work with David Pym)

# EPSRC project

“Semantics of classical proofs”, involving (alphabetically)

Führmann,  
Hyland,  
Pym,  
Robinson,  
Urban.

## The non-determinism of cut-reduction

The proof in the middle (attributed to Lafont) reduces to both  $\Phi_1$  and  $\Phi_2$ :

$$\begin{array}{c} \Phi_1 \\ \vdots \\ \Gamma \vdash \Delta \end{array} \approx \frac{\frac{\frac{\Phi_1}{\vdots} \Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{WR} \quad \frac{\frac{\Phi_2}{\vdots} \Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{WL}}{\Gamma \vdash \Delta} \text{Cut} \approx \begin{array}{c} \Phi_2 \\ \vdots \\ \Gamma \vdash \Delta. \end{array}$$

Therefore, models that preserve meaning along cut reductions are trivial.

## Flawed models

- CCC's with a dualizing object, i.e. an object  $\perp$  such that the map below has an inverse for every object  $A$ .

$$A \longrightarrow ((A \rightarrow \perp) \rightarrow \perp)$$

- Problem: such categories are boolean lattices.
- Translations into classical natural deduction.
  - Problem: admit only the left reduction in Lafont's example (call-by-value) or the right one (call-by-name).

## Overview of this talk

1. Background: categorical models of MLL a'la Cockett & Seely.
2. Symmetric (co)monoids for modelling  $W$  and  $C$ ; *order-enrichment*  $\leq$  such that

$$\Phi \preceq \Psi \implies \mathbf{C}[\Phi] \leq \mathbf{C}[\Psi]$$

(inequational soundness w.r.t. cut reduction). Examples include

- without  $\neg$ :  $(\mathbf{Rel}, \oplus)$ , distributive lattices; free model of proof nets;
- with  $\neg$ :  $(\mathbf{Rel}, \times)$ , boolean lattices; free model of proof nets.

3. Background: Gol for MIX categories (Cockett & Seely).
4. Extension of Gol to  $W$  and  $C$ .

# Models of MLL

# Models of MLL

**Definition 1.** [Cockett/Seely] *Symmetric linearly distributive category:*

- *Symmetric monoidal product  $\otimes$  (for modelling  $\wedge$  and left comma).*
- *Symmetric monoidal product  $\oplus$  (for modelling  $\vee$  and right comma).*
- *Natural transformation satisfying some coherence conditions:*

$$\delta : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C.$$

## Modelling Cut with $\delta$

$$\frac{\begin{array}{c} \Phi \\ \vdots \\ \Gamma \vdash A, \Delta \end{array} \quad \begin{array}{c} \Psi \\ \vdots \\ \Gamma', A \vdash \Delta' \end{array}}{\Gamma', \Gamma \vdash \Delta', \Delta} \text{Cut}$$

is interpreted as

$$\Gamma' \otimes \Gamma \xrightarrow{\Gamma' \otimes [\Phi]} \Gamma' \otimes (A \oplus \Delta) \xrightarrow{\delta} (\Gamma' \otimes A) \oplus \Delta \xrightarrow{[\Psi] \oplus \Delta} \Delta' \oplus \Delta.$$



## Adding negation

To model negation, every object needs a *complement*:

**Definition 2.** A *complement* of an object  $A$  is an object  $A^\perp$  with maps

$$\eta_A : \top \rightarrow (A^\perp \oplus A) \qquad \varepsilon_A : (A \otimes A^\perp) \rightarrow \perp.$$

satisfying two equations.

Fact: every object has (up to isomorphism) at most one complement.

**Definition 3.** An SLDC *with negation* is an SLDC in which every object has a complement.

## Free SLDC generated by proof nets

**Theorem 1. [Cockett/Seely]** *Suppose that*

- *$\mathcal{C}$  is a polygraph,*
- *$Net(\mathcal{C})$  is the polygraph of proof nets generated by  $\mathcal{C}$ , and*
- *$E$  be set of equations between proof nets with the same inputs and outputs.*

*Then  $Net(\mathcal{C})$  modulo  $E$  and certain equations including cut elimination forms the free SLDC generated by  $\mathcal{C}$  and  $E$ .*

Also works in the presence of negation.

Adding weakening and contraction

## Modelling weakening and contraction

- A type-indexed family of symmetric monoids

$$A \oplus A \xrightarrow{\nabla_A} A \xleftarrow{\amalg_A} \perp$$

satisfying the evident coherence conditions.

- A type-indexed family of symmetric co-monoids

$$A \otimes A \xleftarrow{\blacktriangle_A} A \xrightarrow{\langle \rangle_A} \top$$

satisfying the evident coherence conditions.

## Example: associativity

The associativity law of the monoids corresponds to

$$\frac{\frac{\frac{\Phi}{\vdots}}{\Gamma \vdash \Delta_1, (A, A), A, \Delta_2} \text{CR}}{\Gamma \vdash \Delta_1, A, A, \Delta_2} \text{CR}}{\Gamma \vdash \Delta_1, A, \Delta_2} \text{CR} = \frac{\frac{\frac{\Phi}{\vdots}}{\Gamma \vdash \Delta_1, A, (A, A), \Delta_2} \text{CR}}{\Gamma \vdash \Delta_1, A, A, \Delta_2} \text{CR}}{\Gamma \vdash \Delta_1, A, \Delta_2} \text{CR}.$$

## Models of the classical sequent calculus

**Definition 4.** A *Dummett category* is partial-order enriched symmetric linearly distributive category with symmetric monoids and comonoids such that

1.  $\otimes$ , and  $\oplus$  are monotonic in both arguments;
2. parametric versions of the laws below hold.

<i>to model cut reductions involving <math>C</math></i>	<i>to model cut reductions involving <math>W</math></i>
$f \circ \blacktriangledown \leq \blacktriangledown \circ (f \oplus f)$ $\blacktriangle \circ f \leq (f \otimes f) \circ \blacktriangle$	$f \circ \square \leq \square$ $\langle \rangle \circ f \leq \langle \rangle$

A *classical category* is a Dummett category with  $A^\perp \otimes A \longrightarrow \perp$  and  $\top \longrightarrow A \oplus A^\perp$ .

## Examples of Dummett categories

Category	$f \leq g$	negation?	$\otimes = \oplus$ ?
$(\mathbf{Rel}, \times)$	$f \subseteq g$	yes	yes
boolean lattice $\mathbf{B}$	trivial	yes	no
free model built from nets	$f \preceq g$	optional	no
$(\mathbf{Rel}, \uplus)$	$f \supseteq g$	no	yes
$\text{GoI}(\mathbf{Rel}, \uplus)$	see later	yes	yes

## Digression: MIX categories

**Definition 5. [Cockett/Seely]** A *MIX category* is a SLDC together with a morphism

$$m : \perp \longrightarrow \top$$

such that the two evident morphisms

$$A \otimes B \longrightarrow A \oplus B$$

agree.



# Classical logic and MIX

**Theorem 2. [Hasegawa/Führmann]** *Every SLDC with a monoid*

$$\top \oplus \top \xrightarrow{\blacktriangledown} \top \xleftarrow{\boxplus} \perp$$

*is MIX with  $m = \boxplus$ .*

**Corollary 1.** *Every Dummett category is MIX.*

**Remark 1.** *There is also a purely proof-theoretic argument showing that the classical sequent calculus should be MIX under modest assumptions.*

## Hom-semilattices

**Definition 6.** For two morphisms  $f, g : A \longrightarrow B$  of a Dummett category,  $f * g : A \longrightarrow B$  is defined as follows:

$$f * g = A \xrightarrow{\blacktriangle} A \otimes A \begin{array}{l} \nearrow \text{mix} \quad A \oplus A \\ \searrow f \oplus g \\ B \oplus B \end{array} \xrightarrow{\blacktriangledown} B$$

$$\begin{array}{l} \searrow f \otimes g \\ B \otimes B \end{array} \nearrow \text{mix}$$

**Theorem 3.** On every hom-set,  $*$  forms a semilattice (not generally with a unit).

# Equational axiomatization of Dummett categories

**Proposition 1.** *The order  $\leq$  is the one induced by the semilattice  $*$ .*

The *conditional* equalities stating the monotonicity of  $\circ$ ,  $\otimes$ , and  $\oplus$  can be replaced by unconditional equalities. So:

**Theorem 4.** *Dummett categories can be axiomatized as SLDCs with symmetric monoids and symmetric comonoids subject to a number of unconditional equalities.*

## Free classical category generated by nets

**Theorem 5.** *Suppose that*

- *$\mathcal{C}$  is a polygraph containing edges for  $W$  and  $C$ ,*
- *$Net(\mathcal{C})$  is the polygraph of nets generated by  $\mathcal{C}$ , and*
- *$E$  be set of equations between nets with the same inputs and outputs.*

*Then  $Net(\mathcal{C})$  modulo  $E$  and certain (in)equalities including cut elimination forms the free classical category generated by  $\mathcal{C}$  and  $E$ .*

**Proof.** A special case of the free construction by Cockett & Seely, because of the previous theorem.  $\square$

Gol

## Traced objects

**Definition 7.** A *trace* for an object  $U$  of an SLDC  $\mathbf{C}$  is a family of functions

$$\text{tr}_U^{AB} : \mathbf{C}(U \otimes A, U \oplus B) \longrightarrow \mathbf{C}(A, B)$$

satisfying axioms called “Yanking”, “Tightening”, and “Superposing”.

## Trace and compactness

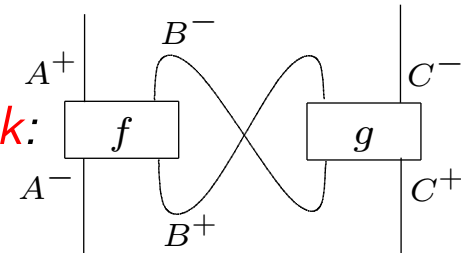
- There is a notion of **compatibility** between a trace on  $U$  and a trace on  $V$ .
- The Gol construction by Cockett & Seely requires a set  $\mathcal{U}$  of pairwise compatible traces objects.
- If  $\mathcal{U}$  contains every object, then the MIX category is **compact**—that is, it is essentially a symmetric monoidal category. The trace is the usual one.

# The GoI construction

**Definition 8.** Given a traced symmetric monoidal category  $\mathbf{C}$ , the category  $\mathbf{GoI}(\mathbf{C})$  is defined as follows:

- Objects are pairs  $(A^+, A^-)$  of objects of  $\mathbf{C}$ ;
- A morphism  $f : (A^+, A^-) \longrightarrow (B^+, B^-)$  of  $\mathbf{GoI}(\mathbf{C})$  is a morphism  $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$  of  $\mathbf{C}$ ;

- Composition is defined by *symmetric feedback*:





## The category $\text{GoI}(\mathbf{C})$

**Theorem 6. [Abramsky/Joyal/Street/Verity]**  $\text{GoI}(\mathbf{C})$  is a compact closed category (= a SLDC with negation such that  $\otimes = \oplus$ ,  $\perp = \top$ , and  $\delta = \alpha$ ).

## The extended GoI construction

**Theorem 7. [Hasegawa]** *If  $\mathbf{C}$  is a traced Dummett category, then  $\text{GoI}(\mathbf{C})$  is a classical category.*

(Generalized version of theorem in our LICS paper.)

**Proof.** The order  $\leq$  is inherited from  $\mathbf{C}$ . The monoids are defined as follows:

$$\blacktriangledown(A^+, A^-) = \begin{array}{c} A^+ \oplus A^+ \quad | \\ \boxed{\blacktriangledown} \\ \quad \quad \quad | \\ \quad \quad \quad \text{---} \\ \quad \quad \quad | \\ \blacktriangle \quad \quad | \\ \boxed{\blacktriangle} \\ A^- \oplus A^- \quad | \\ \quad \quad \quad | \\ \quad \quad \quad A^+ \end{array} \quad \quad \quad \blacksquare(A^+, A^-) = \begin{array}{c} \top \quad | \\ \boxed{\blacksquare} \\ \quad \quad \quad | \\ \quad \quad \quad \text{---} \\ \quad \quad \quad | \\ \diamond \quad \quad | \\ \boxed{\diamond} \\ \top \quad | \\ \quad \quad \quad A^+ \end{array} .$$

Dually for the comonoids. (Checking the inequalities is hard.)  $\square$

## Ongoing work

- More non-compact models. Games? (Pym/Ritter.)
- Extension to predicate logic (McKinley)
- More general models (but without order-enrichment) covering

$$\text{id}_{A \otimes B} \neq \text{id}_A \otimes \text{id}_B \quad \text{id}_{A \oplus B} \neq \text{id}_A \oplus \text{id}_B$$

(Bellin/Hyland/Robinson/Urban).