Equational Lifting Monads

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Abstract

We introduce the notion of an *equational lifting monad*: a commutative strong monad satisfying one additional equation (valid for monads arising from partial map classifiers). We prove that any equational lifting monad has a *representation* by a partial map classifier such that the Kleisli category of the former fully embeds in the partial category of the latter. Thus equational lifting monads precisely capture the (partial) equational properties of partial map classifiers. The representation theorem also provides a tool for transferring non-equational properties of partial map classifiers to equational lifting monads. It is proved using a direct axiomatization of the Kleisli categories of equational lifting monads as *abstract Kleisli categories* with extra structure. This axiomatization is of interest in its own right.

1 Introduction

Ever since Moggi's pioneering work [11,12], the use of strong monads has provided a structural discipline underpinning the categorical approach to denotational semantics. The underlying idea is to make a denotational distinction between the operational notions of *value* and *computation* by modelling them in two separate categories. The category of values, **C**, is a familiar mathematical category of total functions in which the usual datatypes are given their standard universal properties. Programs, however, are interpreted in the category of computations, which is obtained (at least in the call-by-value case) as the Kleisli category of a strong monad on **C** embodying a suitable "notion of computation". The precise nature of the notion of computation varies with

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the programming language modelled. In general it will cater for the various kinds of possible computational effect such as: nontermination, nondeterminism, input/output, . . . (see [12]).

In this paper we concern ourselves solely with one of the simplest forms of computational effect: deterministic computation with possible nontermination. This notion of computation is naturally modelled using partial functions. Thus, in a categorical setting, one looks for monads embodying notions of partiality.

The study of such "lifting" monads goes back to work of Mulry, Rosolini and Moggi in the 1980s [13,17,10]. In particular, in his PhD thesis [17], Rosolini considered a general categorical approach to partiality (based on the associated notions of *dominion*, see Section 2, and *dominance*) and proved representation results into categories of partial functions on presheaf toposes. In computer science, this categorical approach to partiality has proved its value through applications in *axiomatic domain theory* [4,5] and *synthetic domain theory* [7].

However, in spite of the above, there are reasons to look for more general approaches to partiality. In particular, the notion of dominion requires every partial map to have its domain of definition represented as an object of the category. There are cases in which it is dubious that such domain objects should be assumed as primitive. For example, the requirement of their existence prevents one from building syntactic models in which the objects are the types of a programming language and the morphisms are programs (unless either the programs are very simple or the types are very complex). It might be argued that such syntactic models should not lie within the realm of semantics. But one of the elegant features of category theory is that it potentially allows syntax and semantics to be treated on a par, with syntax characterised mathematically in terms of a free category with structure.

In this paper we identify the properties a strong monad must possess in order for its Kleisli category to behave like an induced category of partial maps. The standard examples here are the monads determined by partial map classifiers with respect to a dominion — we review the properties of such *dominical lifting monads* in Section 2. However, as motivated above, we aim for an axiomatization of such "lifting" monads without making reference to any notion of domain. The domain-free axiomatization will explicitly allow the class of models to include natural examples (such as the term models mentioned above) which are otherwise excluded.

Section 3 presents the main contribution of this paper, the notion of *equational lifting monad* and the justification of it. Equational lifting monads have a simple equational axiomatization. Via a representation theorem (Theorem 10), we show that (the Kleisli categories of) equational lifting monads have the same equational properties as (the Kleisli categories of) *dominical* lifting monads (those arising from dominions). The theorem also shows that equational lifting monads inherit some useful non-equational properties from dominical lifting monads (Corollary 12). Other non-equational properties do not hold in general (Corollary 13), but conditions uder which they do are given by a strengthened representation theorem (Corollary 11).

The proof of Theorem 10 occupies the remainder of the paper. However, much of the development directed towards its proof is of independent interest.

In Sections 4 and 5 we provide an alternative perspective on equational lifting monads by giving a direct axiomatization of the categorical structure of their Kleisli categories. This work continues in a tradition, exemplified by Robinson and Rosolini's notion of *p-category* [16], of providing direct, domain-free axiomatizations of categories of partial maps (see [16] for other references). In the case of p-categories, the axiomatized categories correpond to categories of partial maps with a suitable product structure. The structure we require is that of a p-category that, in addition, is the Kleisli category of an equational lifting monad on its associated total category. Our axiomatization of such categories is obtained by extending the notion of an *abstract Kleisli category* [6], which provides a direct axiomatization of categories that arise as Kleisli categories, with the necessary additional structure. We call the axiomatized categories *abstract Kleisli p-categories*.

In Section 6 we characterise when an abstract Kleisli p-category arises as the Kleisli category of a dominical lifting monad (Theorem 34). The characterisation, similar to [16], requires a collection of idempotents in the Kleisli category to split. We then show that the structure of an abstract Kleisli p-category is preserved under a formal idempotent splitting, allowing any abstract Kleisli p-category to be embedded in a dominical one. The proof is surprisingly involved.

Finally, in Section 7, we complete the proof of Theorem 10, making crucial use of the results and constructions from the previous sections. We also discuss some other miscellaneous properties of equational lifting monads.

2 Review of lifting monads

In this section we review the standard categorical approach to partiality and the notion lifting monad it induces. All the definitions and results are contained (at least implicitly) in [17].

Definition 1 A *dominion* on a category **C** is given by a collection **D** of monomorphisms in **C** that is closed under composition, contains every isomorphism, and is closed under pullback along arbitrary morphisms.

Let **C** be a category and let **D** be a dominion on it. In general we use the symbol $\hookrightarrow$ to represent monos in **D**.

A *(***D***-)partial map* from an object A of **C** to an object B is given by an equivalence class of spans of the form:

$$\begin{array}{ccc} A' & \xrightarrow{f} & B \\ {\scriptstyle m}\big\downarrow & & \\ A & & \end{array} \tag{1}$$

where m is in **D**. (The notion of equivalence between spans is obvious.) The conditions imposed on **D** are just what is needed for the collection of partial maps to form a category (with the same objects as **C**). In particular, composition is performed using closure under pullbacks. We write $\mathbf{Ptl}_{\mathbf{D}}(\mathbf{C})$ for the category of partial maps (often omitting **D**). There is an evident faithful functor $J : \mathbf{C} \to \mathbf{Ptl}(\mathbf{C})$ which is the identity on objects.

Definition 2 We say that **C** has *(***D***-)partial map classifiers* if, for every object B there exists a map $B \xrightarrow{\eta} LB$ in **D** such that, for every partial map (as in (1) above), there exists a unique *characteristic map* $A \xrightarrow{g} LB$ such that the square below is a pullback.

$$\begin{array}{ccc} A' & \xrightarrow{f} & B \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta} \\ A & \xrightarrow{g} & LB \end{array}$$

Proposition 3 *The following are equivalent.*

(1) **C** *has partial map classifiers.*
(2) The functor J has a right adjoint $K : \mathbf{Ptl}(\mathbf{C}) \to \mathbf{C}$.

When such a right adjoint K exists, its action on objects can always be taken to be $A \mapsto LA$.

Henceforth in this section, we assume that **C** has partial map classifiers and examine some of the consequences.

As with any pair of adjoint functors, the composite functor $L = KJ : \mathbf{C} \to \mathbf{C}$ is a monad. The unit is given by the family $A \xrightarrow{\eta} LA$ already identified. The multiplication, $L^2A \xrightarrow{\mu} LA$, is obtained as the unique map making the square below a pullback:

$$
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
{\scriptstyle \eta \circ \eta}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta} \\
L^2A & \xrightarrow{\mu} & LA
\end{array}
\tag{2}
$$

Moreover, as the functor $J : \mathbf{C} \to \mathbf{Ptl}(\mathbf{C})$ is an isomorphism on objects, the category $\mathbf{Ptl}(\mathbf{C})$ of partial maps is (isomorphic to) the Kleisli category of the monad.

Monads that are obtained, as above, from partial map classifiers will be our paradigmatic examples of "lifting" monads representing partiality. For later purposes, it will be handy to have a name for them.

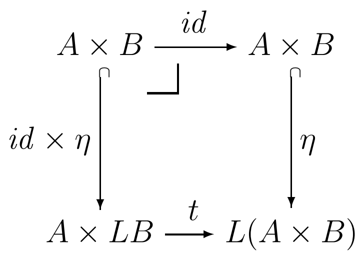
Definition 4 We say that a monad (L, η, μ) on a category $\mathbf{C}$ is a *dominical lifting monad* if there exists a dominion $\mathbf{D}$ on $\mathbf{C}$ such that $\mathbf{C}$ has $\mathbf{D}$-partial map classifiers and (L, η, μ) is the monad determined by the adjunction between $\mathbf{C}$ and $\mathbf{Ptl_D}(\mathbf{C})$.

Dominical lifting monads have many properties not shared by arbitrary monads. For example, the natural transformations η and μ are both *cartesian*, i.e. their naturality squares below are always pullbacks.

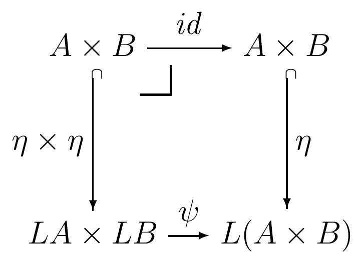
$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \eta}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta} \\
LA & \xrightarrow{Lf} & LB
\end{array}
\qquad\qquad
\begin{array}{ccc}
L^2A & \xrightarrow{L^2f} & L^2B \\
{\scriptstyle \mu}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mu} \\
LA & \xrightarrow{Lf} & LB
\end{array}
\tag{3}
$$

In addition, the functor L preserves these pullback squares. Indeed it preserves *all* pullbacks.

Now suppose $\mathbf{C}$ has finite products and let (L, η, μ) be a dominical lifting monad. Define $A \times LB \xrightarrow{t} L(A \times B)$ as the unique map making the square below into a pullback.



Then $A \times LB \xrightarrow{t} L(A \times B)$ is natural in A and B and provides a (unique) *strength* for the monad in the sense of Kock [9] (see also [11,12]). The *costrength* $LA \times B \xrightarrow{t'} L(A \times B)$ can be defined analogously. A related symmetric map $LA \times LB \xrightarrow{\psi} L(A \times B)$ is obtained as the unique morphism making the square below into a pullback.

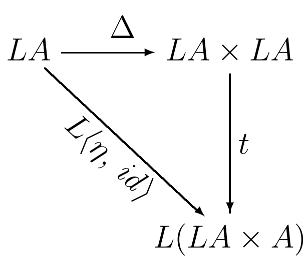


By pasting pullback squares, it is straightforward to verify that

$$\mu \circ Lt' \circ t \quad = \quad \psi \quad = \quad \mu \circ Lt \circ t' \;, \tag{4}$$

i.e. the monad is *commutative* in the sense of Kock [9].

We end this section by noting that the following diagram always commutes for dominical lifting monads



(5)

(where $LA \xrightarrow{\Delta} LA \times LA$ is the diagonal of the product). This holds because the diagrams below are both pullbacks (the second because η is cartesian).

$$
\begin{array}{ccccc}
A & \xrightarrow{\langle \eta,\, id \rangle} & LA \times A & \xrightarrow{id} & LA \times A \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle id \times \eta} & & \downarrow{\scriptstyle \eta} \\
LA & \xrightarrow{\Delta} & LA \times LA & \xrightarrow{t} & L(LA \times A)
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\langle \eta,\, id \rangle} & LA \times A \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta} \\
LA & \xrightarrow{L\langle \eta,\, id \rangle} & L(LA \times A)
\end{array}
$$

So, by the uniqueness of characteristic maps, $L\langle \eta,\, id \rangle = t \circ \Delta$. The diagram (5) will be very important in the sequel.

3 Equational lifting monads

In the previous section we investigated some of the special properties of dominical lifting monads. In this section we aim to precisely identify their equational properties.

We begin with a very brief resumé on strong monads. Let $\mathbf{C}$ be a category with specified finite products. A *strong monad* on $\mathbf{C}$ is a monad (L, η, μ) together with a natural strength $A \times LB \xrightarrow{t} L(A \times B)$ satisfying the equations in [9,11,12]. The dual $LA \times B \xrightarrow{t'} L(A \times B)$ is defined by $t' = Lc \circ t \circ c$ (where c is the symmetry of product). A strong monad is *commutative* if $\mu \circ Lt' \circ t = \mu \circ Lt \circ t' : LA \times LB \longrightarrow L(A \times B)$, in which case we *define* a natural transformation $LA \times LB \xrightarrow{\psi} L(A \times B)$ by equation (4). (Note that t and μ are now primitive data rather than defined entities.)

We say that a monad (L, η, μ) satisfies the *mono requirement* if all components $A \xrightarrow{\eta} LA$ are mono and that it satisfies the *equaliser requirement* if $A \xrightarrow{\eta} LA \underset{L\eta}{\overset{\eta}{\rightrightarrows}} L^2A$ is an equaliser diagram. (The terminology is due to Moggi [11]. A monad satisfying the equaliser requirement is also said to be of *descent type* [1].) The mono requirement is easily seen to be equivalent to faithfulness of L and also to the faithfulness of the right-adjoint functor J from $\mathbf{C}$ to the Kleisli category $\mathbf{C}_L$. The equaliser requirement clearly implies the mono requirement. (The converse also holds under some circumstances, e.g. if $\mathbf{C}$ is an elementary topos [1].)

We now give the main definition of the paper.

Definition 5 We say that a strong monad (L, η, μ, t) is an *equational lifting monad* if it is commutative and also $L\langle \eta,\, id \rangle = t \circ \Delta : LA \longrightarrow L(LA \times A)$.

The characteristic equation of an equational lifting monad is just diagram (5) from Section 2 (except that η and t are now primitive data). The reader is encouraged to understand the meaning of this equation by working out the interpretation of the two terms for familiar (not necessarily commutative) monads on the category of sets, e.g. powerset (for which it fails) and exceptions, $(-)+E$, (for which it holds). We remark that equation (5) is somewhat similar to the *Euclidean pinciple* recently introduced by Taylor [18] as part of a characterisation of dominances.

In the previous section we established that every dominical lifting monad is an equational lifting monad. The goal of this paper is twofold: to show that equational lifting monads possess all the equational properties of dominical lifting monads; and to characterise the precise conditions under which the other (non-equational) properties of dominical lifting monads hold also for equational lifting monads.

We start with a simple proposition establishing two basic equational properties of equational lifting monads.

Proposition 6 *For an equational lifting monad (L, η, μ, t), the diagrams below commute.*

$$
\begin{array}{ccc}
LA & \xrightarrow{\Delta} & LA \times LA \\
 & {}_{L\Delta}\searrow & \downarrow \psi \\
 & & L(A \times A)
\end{array}
\qquad\qquad
\begin{array}{ccc}
L^2A & \xrightarrow{id} & L^2A \\
\langle \mu,\, L! \rangle \downarrow & & \uparrow L(\pi_1) \\
LA \times L1 & \xrightarrow{t} & L(LA \times 1)
\end{array}
\qquad (6)
$$

Proof. The first equality is easily proved from equation (5) and properties of a strong monad, by expanding the definition $\psi = \mu \circ L(t') \circ t$.

For the second diagram, using (5) and the naturality of t, one obtains:

$$
\begin{aligned}
id_{L^2A} &= L(\mu) \circ L(\eta) \\
&= L(\mu) \circ L(\pi_{LA}) \circ L(\langle \eta,\, id \rangle) \\
&= L(\pi_1) \circ L(\mu \times !) \circ t \circ \Delta \\
&= L(\pi_1) \circ t \circ (\mu \times L!) \circ \Delta \\
&= L(\pi_1) \circ t \circ \langle \mu,\, L! \rangle
\end{aligned}
$$

The left-hand diagram above expresses that any equational lifting monad is *relevant* in the sense of Jacobs [8]. (Not every commutative relevant monad is an equational lifting monad. A simple counterexample is the $(-)^2$ monad on the category of sets.)

The right-hand diagram has a couple of interesting consequences. One easy consequence is equation (7) of [2]: $L(\pi_1) \circ t \circ \langle id,\, L! \rangle = L\eta : LA \longrightarrow L^2 A$, which was used there as part of a non-equational axiomatization of lifting monads. Another consequence is that the arrows $L1 \xleftarrow{L!} L^2 A \xrightarrow{\mu} LA$ are jointly monic (because $\langle \mu,\, L! \rangle$ is a split mono).

Our main theorem will state that every equational lifting monad can be sufficiently well "represented" by a dominical lifting monad. To state this we require morphisms of strong monads. For convenience, we work with a strict version (which is perfectly sufficient for our needs).

Definition 7 Suppose (L, η, μ, t) is a strong monad on $\mathbf{C}$ and (L', η', μ', t') is a strong monad on $\mathbf{C}'$. We say a functor $F : \mathbf{C} \to \mathbf{C}'$ is *strong monad preserving* if the strong monad data is strictly preserved.

Observe that a strong monad preserving functor $F : \mathbf{C} \to \mathbf{C}'$ determines (in the obvious way) a functor $F_K : \mathbf{C}_L \to \mathbf{C}'_{L'}$ between the Kleisli categories of L and L' (we shall often refer to strong monads by just naming the underlying functors). We say that F itself is *Kleisli full* (resp. *Kleisli faithful*) to mean that the induced functor F_K is full (resp. faithful). Observe that full and faithful imply Kleisli full and Kleisli faithful respectively. The converse implications are closely linked to properties of η.

Proposition 8 *Suppose L is a monad on $\mathbf{C}$, L' is a monad on $\mathbf{C}'$ and $F : \mathbf{C} \to \mathbf{C}'$ is monad preserving.*

(1) If the mono requirement holds for L' and F is Kleisli faithful then the mono requirement holds for L if and only if F is faithful.
(2) If the equaliser requirement holds for L' and F is Kleisli full and Kleisli faithful then the equaliser requirement holds for L if and only if F is full and faithful.

Proof.

(1) Let J' be the embedding $\mathbf{C}' \longrightarrow \mathbf{C}'_{L'}$. As the hypothesis on L' amounts to J' being faithful, the equivalence follows from the equality $F_K J = J' F$, as it implies that F is faithful if and only if J is.
(2) For the left-to right implication, the faithfulness of F follows from *1*. To prove fullness, take $f \in \mathbf{C}'(FA, FB)$ and show that there exists $g \in \mathbf{C}(A, B)$ such that $Fg = f$. First consider the composition $\eta' \circ f \in \mathbf{C}'_K(FA, FB)$. As F_K is full, there exists $h \in \mathbf{C}_K(A, B)$ such that $Fh = \eta' \circ f$. Now we show that $L\eta \circ h = \eta \circ h$ and apply the hypothesis that

L satisfies the equaliser requirement. This equality is easily verified by applying the faithful functor F_K:

$$F(L\eta) \circ Fh = (L'\eta') \circ \eta' \circ f = \eta' \circ \eta' \circ f = F\eta \circ Fh.$$

So, as L satisfies the equaliser requirement, there exists $g \in \mathbf{C}(A, B)$ such that $\eta \circ g = h$. Finally

$$\eta' \circ Fg = F\eta \circ Fg = Fh = \eta' \circ f$$

and so, as η' is mono, $Fg = f$.

To show the other implication consider $f \in \mathbf{C}(A, LB)$ such that $\eta \circ f = (L\eta) \circ f$. Then $\eta' \circ Ff = (L'\eta') \circ Ff$. So, as L' satisfies the equaliser requirement, there exists $h \in \mathbf{C}'(FA, FB)$ such that $Ff = \eta' \circ h$. Now, since F is full, there exists $g \in \mathbf{C}(A, B)$ such that $Fg = h$. So we obtain

$$Ff = \eta' \circ Fg = (F\eta) \circ (Fg) = F(\eta \circ g).$$

As F is faithful, this implies $f = \eta \circ g$. The uniqueness follows from the fact that η is mono, as η' is mono and F is faithful.

Definition 9 Let L be a strong monad on $\mathbf{C}$. A *dominical representation* of L is given by a category $\mathbf{C}'$ with finite products and dominical lifting monad L' together with a strong monad preserving functor $F : \mathbf{C} \to \mathbf{C}'$.

We refer to a representation as being (Kleisli) full/faithful if the property mentioned holds of the representing functor F.

Both the commutativity equation (4) and diagram (5) can be viewed as equations between Kleisli category morphisms. Also, these equations hold in any dominical lifting monad. It follows that any strong monad with a Kleisli faithful dominical representation is necessarily an equational lifting monad. Our principal theorem gives a strengthened converse to this observation.

Theorem 10 *If L is an equational lifting monad then L has a dominical representation that is Kleisli full and Kleisli faithful.*

Given the theorem, conditions for obtaining strengthened representations are now immediate from Proposition 8.

Corollary 11 *Let L be an equational lifting monad.*

(1) L has a dominical representation that is faithful and Kleisli full if and only if it satisfies the mono requirement.
(2) L has a dominical representation that is full and faithful if and only if it satisfies the equaliser requirement.

The proof of Theorem 10 will be given in Section 7, after preparatory work occupying the remainder of the paper.

It is worth explaining the significance of Theorem 10 and Corollary 11. The notion of dominical lifting monad corresponds to an accepted categorical notion of partiality, with the associated category of partial maps obtained as the Kleisli category. Our aim is to establish that, for equational lifting monads, the Kleisli category also acts just like a category of partial maps determined by a dominical lifting monad. Theorem 10 states in what sense this is indeed the case. Corollary 11 concerns relating the base category of the equational lifting monad with that of the dominical lifting monad.

We end this section with some applications of Theorem 10 and Corollary 11, illustrating how they can be used to establish non-equational properties of equational lifting monads. Recall the definition of a *cartesian* natural transformation (see the text after Definition 4).

Corollary 12 *For any equational lifting monad, μ is cartesian.*

Proof. It suffices to apply the representing functor F to the right-hand diagram of (3) and note that all arrows involved are actually arrows in the Kleisli category $\mathbf{C}'_{L'}$. So the fullness and faithfulness of F_K and the cartesianness of μ' (the monad multiplication in $\mathbf{C}'_{L'}$) together imply the result.

Incidentally, we also have a direct equational proof that μ is cartesian. Given a cone $LA \xleftarrow{g} C \xrightarrow{h} L^2B$ (to the span in the right-hand diagram of (3)), the universal map is $L(\pi_1) \circ t \circ \langle g, h\rangle : C \longrightarrow L^2A$. Our verification that the necessary equalities hold is extraordinarily long, so it seems preferable to defer to the proof provided by Corollary 12.

Theorem 10 can be used in an analogous way to transfer other pullback properties of dominical lifting monads to equational lifting monads, for example, to establish that diagram (2) is also a pullback for any equational lifting monad. However, not all pullbacks transfer automatically.

Corollary 13 *For any equational lifting monad, η is a cartesian natural transformation if and only if the equaliser requirement is satisfied.*

Proof. The left-to-right implication is easily proved for an arbitrary monad. For the right-to-left implication, assume the equaliser requirement holds. By Corollary 11, L has a full and faithful dominical representation by a functor $F : \mathbf{C} \to \mathbf{C}'$. It suffices to apply the F to the left-hand diagram of (3), and use the cartesianness of η' in $\mathbf{C}'$ to infer that it is cartesian in $\mathbf{C}$.

4 Abstract Kleisli categories

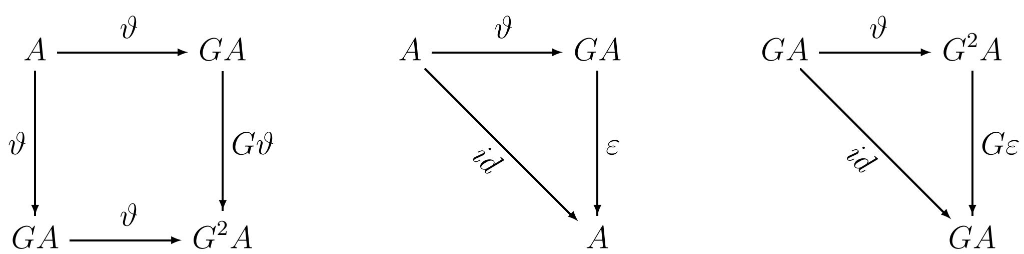
In the previous two sections the category $\mathbf{C}$ together with its monad was taken as primitive, and the notion of equational lifting monad was defined in order to capture the idea of when the Kleisli category is an interesting category of partial maps. On occasion, however, it is arguably more natural to consider the partial category itself as the primitive category. For example, in the case of a category with partial functions and associated product structure, this was the approach adopted by Robinson and Rosolini with their *p-categories* [16].

In the next two sections we provide such a direct axiomatization of the Kleisli categories of equational lifting monads, which are our partial categories of interest. The axiomatization will be obtained by extending Führmann's *abstract Kleisli categories* [6], which directly axiomatize the structure of Kleisli categories of monads. In this section we review the definitions and results that we shall need from [6], in the special case of a *commutative precartesian abstract Kleisli category*, which is of most relevance to us. In Section 5 we shall extend the axiomatization with with an equation corresponding to the additional properties of equational lifting monads, to give us the notion of an *abstract Kleisli p-category*.

Definition 14 An *abstract Kleisli category* is a category $\mathbf{K}$ together with:

(1) A functor $G : \mathbf{K} \to \mathbf{K}$;
(2) A transformation[4] $A \xrightarrow{\vartheta_A} GA$ (called *thunk*);
(3) A natural transformation $GA \xrightarrow{\varepsilon_A} A$ (called *force*);

such that $\vartheta_G : G \to G^2$ is a natural transformation, and the following diagrams commute:



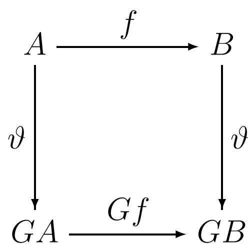
Given any category $\mathbf{C}$ with a monad L, the Kleisli category $\mathbf{C}_L$ forms an abstract Kleisli category. The endofunctor $G : \mathbf{C}_L \to \mathbf{C}_L$ is obtained as the composite $\mathbf{C}_L \xrightarrow{K} \mathbf{C} \xrightarrow{J} \mathbf{C}_L$ around the adjunction determined by the monad. Thus on objects we have $GA = LA$. The thunk morphism $A \xrightarrow{\vartheta} GA$ in $\mathbf{C}_L$ is given by $A \xrightarrow{\eta \circ \eta} L^2 A$ in $\mathbf{C}$. The force map $GA \xrightarrow{\varepsilon} A$ in $\mathbf{C}_L$ is just the

[4] By transformation we mean an arbitrary family of arrows indexed by objects.

counit of the adjunction, which is explicitly given by the identity $LA \xrightarrow{id} LA$ in $\mathbf{C}$.

Conversely, an abstract Kleisli category $\mathbf{K}$ suffices to determine a category $\mathbf{C}$ with a monad L such that $\mathbf{K}$ is isomorphic to $\mathbf{C}_L$ with G, ϑ, and ε induced as above.

Definition 15 In an abstract Kleisli category $\mathbf{K}$, a morphism $A \xrightarrow{f} B$ is said to be *thunkable* if the diagram below commutes:



The collection of thunkable maps forms a subcategory of $\mathbf{K}$. We write $\mathbf{K}_t$ for this subcategory and $J : \mathbf{K}_t \to \mathbf{K}$ for the inclusion functor.

Proposition 16 *Given an abstract Kleisli category* $\mathbf{K}$*, the inclusion functor* $J : \mathbf{K}_t \to \mathbf{K}$ *has a right adjoint, inducing a monad* L *on* $\mathbf{K}_t$ *such that* $\mathbf{K}$ *is isomorphic to the Kleisli category of* L *with* G*,* ϑ*, and* ε *determined as above. Moreover,* L *satisfies the equaliser requirement.*

The proof of the proposition can be found in [6]. Here, we just identify the adjunction between $\mathbf{K}$ and $\mathbf{K}_t$. For each $f \in \mathbf{K}(A, B)$, define

$$[f] = Gf \circ \vartheta \tag{7}$$

This determinines an adjunction

$$[-] : \mathbf{K}(A, B) \cong \mathbf{K}_t(A, GB) \tag{8}$$

where A ranges over $\mathbf{K}_t$ and B over ranges $\mathbf{K}$, with unit ϑ and counit ε.

The above constructions map any category and monad $(\mathbf{C}, L, \eta, \mu)$ to an induced abstract Kleisli category $(\mathbf{C}_L, G_L, \vartheta_L, \varepsilon_L)$, and map any abstract Kleisli category $(\mathbf{K}, G, \vartheta, \varepsilon)$ to an induced category $\mathbf{K}_t$ with a monad satisfying the equaliser requirement. Both operations are functorial in an appropriate sense, and together they establish a reflection from categories with monads to abstract Kleisli categories [6]. We only need one part of this reflection.

Suppose we start with a category $\mathbf{C}$ with monad L. We write $(\mathbf{C}_L)_t$ for the category obtained by extracting the thunkable maps from the Kleisli category construed as an abstract Kleisli category. The unit of the reflection of [6] is given by the evident functor $U : \mathbf{C} \to (\mathbf{C}_L)_t$ (specifically U maps a morphism $A \xrightarrow{f} B$ in $\mathbf{C}$ to the thunkable morphism $A \xrightarrow{\eta\circ f} B$ in $\mathbf{C}_L$). Moreover, U is an isomorphism of categories if and only if L satisfies the equaliser requirement.

The construction given above maps any monad to one satisfying the equaliser requirement. To apply this construction to equational lifting monads, we also need to ensure that strength, commutativity and equation (5) are all preserved.

In [6] the notion of *precartesian abstract Kleisli category* is introduced to extend the reflection theorem to strong monads. In this paper, we are interested only in commutative strong monads. This allows some of the complications of general strong monads to be avoided (in particular, we can work with a monoidal structure rather than a premonoidal structure).

Definition 17 A *commutative precartesian abstract Kleisli category* is given by an abstract Kleisli category $\mathbf{K}$ together with a symmetric monoidal functor $\otimes$ with unit I such that $\mathbf{K}_t$ has a distinguished symmetric monoidal structure given by finite products and the functor $J : \mathbf{K}_t \to \mathbf{K}$ strictly preserves the symmetric monoidal structure.

Given a category $\mathbf{C}$ with finite products and a commutative strong monad L, the induced abstract Kleisli category $\mathbf{C}_L$ forms a commutative precartesian abstract Kleisli category, with $\otimes$ on $\mathbf{C}_L$ generated by the product on $\mathbf{C}$ (see e.g. [8]). Conversely, given a commutative precartesian abstract Kleisli category $\mathbf{K}$, the inclusion $J : \mathbf{K}_t \to \mathbf{K}$ has a right adjoint determining a strong monad on $\mathbf{K}_t$ such that $\mathbf{K}$ is the induced commutative precartesian Kleisli category (up to isomorphism). Once again, the constructions can be combined to obtain, for any category $\mathbf{C}$ with commutative strong monad, a category $(\mathbf{C}_L)_t$ with commutative strong monad satisfying the equaliser requirement, and a strong monad preserving functor $U : \mathbf{C} \to (\mathbf{C}_L)_t$. Note that $(\mathbf{C}_L)_t$ and U are constructed exactly as in the case without strength, so Proposition 16 applies *mutatis mutandis.*

The definition of commutative abstract Kleisli category is compact, but is not as direct as one would like as it involves identifying structure on the derived category $\mathbf{K}_t$. Accordingly, we now state an alternative equational characterisation of commutative precartesian abstract Kleisli categories, whose proof (in the, more general, non commutative case) can be found in [6]. The characterisation will be very helpful in Sections 5 and 6.

Definition 18 Suppose that $\mathbf{K}$ is an abstract Kleisli category with a functor $\otimes : \mathbf{K}\times\mathbf{K} \to \mathbf{K}$, an object I, and transformations $A \xrightarrow{\Delta} A\otimes A$ and $A \xrightarrow{!} I$. A morphism $A \xrightarrow{f} B$ of $\mathbf{K}$ is called *copyable* if the left-hand diagram below commutes. It is called *discardable* if the right-hand diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes f} \\ B & \xrightarrow{\Delta} & B \otimes B \end{array} \qquad\qquad \begin{array}{ccc} A & & \\ {\scriptstyle f}\downarrow & \searrow{\scriptstyle !} & \\ B & \xrightarrow{!} & I \end{array} \tag{9}$$

Observe that any (commutative) precartesian abstract Kleisli category provides the functor and transformations required by Definition 9, as they are inherited from the cartesian structure on $\mathbf{K}_t$. Moreover, it follows that every thunkable morphism is both copyable and discardable. Neither of the converses holds in general. See [6] for further discussion.

In order to formulate the next proposition, and for many future calculations, it is convenient to define a pairing operation on categories carrying the structure identified in Definition 9. For morphisms $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$ in such a $\mathbf{K}$, define the pairing $A \xrightarrow{\langle f, g\rangle} B \otimes C$ to be the composite:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes C$$

It is important to note that the equalities $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$ do *not* hold in general.

Proposition 19 ([6]) *Suppose that* $\mathbf{K}$ *is an abstract Kleisli category, with a functor* $\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$*, an object* I*, and transformations:*

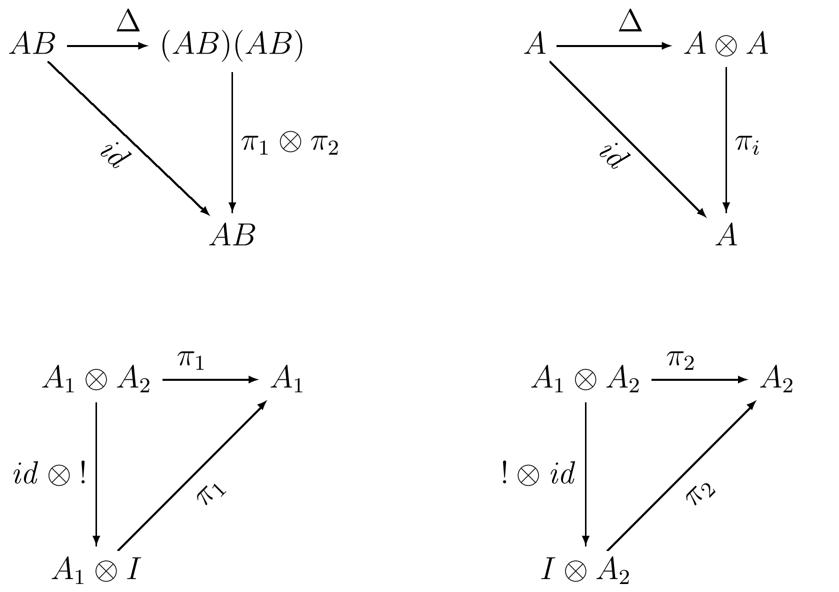
$$A \xrightarrow{\Delta} A \otimes A \qquad A_1 \otimes A_2 \xrightarrow{\pi_i} A_i \qquad A \xrightarrow{!} I$$

Then $\mathbf{K}$ *determines a commutative precartesian abstract Kleisli category if and only if:*

(1) All morphisms of the form $[f]$ *are copyable and discardable.*
(2) All components of Δ*,* π_i*, and* $!$*, as well as all morphisms of the form* $A \otimes [f]$ *and* $[f] \otimes A$*, are thunkable.*
(3) The transformations below are natural.

$$\begin{aligned} \pi_1 &: A \otimes I \to A \\ \pi_2 &: I \otimes A \to A \\ \langle \pi_2, \pi_1\rangle &: A \otimes B \to B \otimes A \\ \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2\rangle\rangle &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \end{aligned}$$

(4) The diagrams below commute (where we write AB for $A \otimes B$).



5 Abstract Kleisli p-categories

We have now given a direct axiomatization of the Kleisli categories of commutative strong monads. It remains to deal with the additional equation (5) of an equational lifting monad. In this section we shall incorporate this equation to give the notion of *abstract Kleisli p-category*, and invesigate its properties.

As the name suggests, it will be the case that our axiomatized abstract Kleisli p-categories are all p-categories in the sense of [16]. To properly understand what follows, it is instructive to begin by considering what additional properties are required by a commutative precartesian abstract Kleisli category for it to simultaneously be a p-category. The discussion will explicitly refer to the axiomatization of p-categories in [16, p.101], which we do not wish to repeat here, as it is not necessary for our main line of development. It should be possible to follow the gist of the discussion even without a copy of [16] to hand.

Let $\mathbf{K}$ be a commutative precartesian abstract Kleisli category. This structure provides the functor $\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ and transformations $A \xrightarrow{\Delta} A \otimes A$ and $A_1 \otimes A_2 \xrightarrow{\pi_i} A_i$ (as in Proposition 19) required by a p-category. The six equations in the definition of p-category involving the projections and the diagonal hold because the transformations are inherited from the finite products on the category of thunkable maps. It remains to check the naturality requirements. The naturality of the associativity and the twist map follows from Proposition 19. That the projections are natural in the non-discarded argument follows from the handy lemma below.

Lemma 20 *In any commutative precartesian abstract Kleisli category, for morphisms* $A \xrightarrow{f} A'$ *and* $B \xrightarrow{g} B'$*, we have*

$$\begin{aligned} \pi_1 \circ (f \otimes g) &= f \circ \pi_1 \circ (id \otimes g) \\ \pi_2 \circ (f \otimes g) &= g \circ \pi_2 \circ (f \otimes id) \end{aligned}$$

(The point here is that f *and* g *don't have to be thunkable).*

Proof.

$$\begin{aligned} \pi_1 \circ (f \otimes g) &= \pi_1 \circ (A' \otimes !) \circ (f \otimes g) \\ &= \pi_1 \circ (f \otimes I) \circ (A \otimes !) \circ (A \otimes g) \\ &= f \circ \pi_1 \circ (A \otimes !) \circ (A \otimes g) \quad \text{because } \pi_1 : A \otimes I \to A \text{ is the monoidal structural iso} \\ &= f \circ \pi_1 \circ (A \otimes g) \end{aligned}$$

To obtain a p-category, there is only one condition left to verify: the naturality of $A \xrightarrow{\Delta} A \otimes A$. This amounts to the copyability of every morphism in $\mathbf{K}$, something that does *not* hold in general. Thus we have arrived at the following characterisation.

Proposition 21 *For a commutative precartesian abstract Kleisli category* $\mathbf{K}$*, the following are equivalent:*

(1) $\mathbf{K}$ *is a p-category (via its commutative precartesian structure).*
(2) Every morphism in $\mathbf{K}$ *is copyable.*

From the above proposition, one might expect to define an "abstract Kleisli p-category" as a commutative precartesian abstract Kleisli category in which every morphism is copyable. Such categories are exactly the Kleisli categories of commutative *relevant* monads in the sense of [8], see the discussion after Proposition 6. However, in that discussion, the $(-)^2$ monad is cited as a commutative relevant monad that is not an equational lifting monad. Thus there is a mismatch between the candidate notion of "abstract Kleisli p-category" and the Kleisli categories of equational lifting monads. This mismatch does not indicate a problem with the notion of equational lifting monad, but rather a subtlety we have overlooked in the suggested definition of "abstract Kleisli p-category". We now illustrate the problem in detail.

For any commutative precartesian abstract Kleisli category **K**, we can associate a *domain* map $A \xrightarrow{\overline{f}} A$ to any map $A \xrightarrow{f} B$, by defining:

$$\overline{f} = A \xrightarrow{\langle id, f\rangle} A \otimes B \xrightarrow{\pi_1} A$$

In the case that **K** is a category of partial maps, we think of $\overline{f}$ as representing the partial function that is defined exactly when f is defined, and which acts as the identity on its domain of definition. However, the definition makes sense in any commutative precartesian abstract Kleisli category.

From the intuition behind the definition of domain maps, there is a natural associated notion of total morphism.

Definition 22 A morphism $A \xrightarrow{f} B$ in **K** is said to be *p-total* if $\overline{f} = id_A$.

Again, although the definition is motivated through considering the case of partial maps, the definition makes sense for any commutative precartesian abstract Kleisli category. Indeed, it turns out that the notion of p-totality is a familiar one in disguise.

Proposition 23 *If* **K** *is a commutative precartesian abstract Kleisli category then a morphism is p-total if and only if it is discardable.*

Proof. Suppose f is p-total. Then:

$$\begin{aligned} !_A &= !_A \circ \overline{f} \quad \text{(because } f \text{ is p-total)} \\ &= !_A \circ \pi_1 \circ (id_A \otimes f) \circ \Delta \\ &= !_B \circ \pi_2 \circ (id_A \otimes f) \circ \Delta \quad \text{(by the product structure on } \mathbf{K}_t) \\ &= !_B \circ f \circ \pi_2 \circ \Delta \quad \text{(by Lemma 20)} \\ &= !_B \circ f \end{aligned}$$

Conversely, suppose f is discardable, then:

$$\begin{aligned} \overline{f} &= \pi_1 \circ (id_A \otimes f) \circ \Delta \\ &= \pi_1 \circ (id_A \otimes !_B) \circ (id_A \otimes f) \circ \Delta \quad \text{(by the product structure on } \mathbf{K}_t) \\ &= \pi_1 \circ (id_A \otimes !_A) \circ \Delta \quad \text{(because } f \text{ is discardable)} \\ &= id_A \quad \text{(by the product structure on } \mathbf{K}_t) \end{aligned}$$

It is an immediate consequence that the p-total maps of **K** form a subcategory of **K**.

The definition of domain maps is taken from [16] (although our notation is different). Although we have defined domain maps for an arbitrary commutative precartesian abstract Kleisli category, they have the expected properties only when **K** is also a p-category. In fact, the properties below all hold for domain maps in an arbitrary p-categories [16]. The first proposition says that the domain operation provides a *restriction structure* in the sense of [3]. The second contains results easily derivable from the first. These results, whose proofs are all easy, will be useful later on.

Proposition 24 *If* **K** *is a commutative precartesian abstract Kleisli category in which every morphism is copyable then the following properties hold (whenever the compositions make sense).*

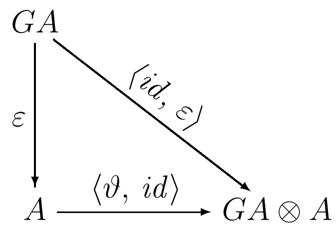
(1) $f \circ \overline{f} = f$
(2) $\overline{g} \circ \overline{f} = \overline{f} \circ \overline{g}$
(3) $\overline{g \circ \overline{f}} = \overline{g} \circ \overline{f}$
(4) $\overline{g} \circ f = f \circ \overline{g \circ f}$

Corollary 25 *If* **K** *is a commutative precartesian abstract Kleisli category in which every morphism is copyable then:*

(1) $\overline{f}$ *is idempotent;*
(2) if f *is mono then* f *is p-total;*

The subtlety referred to above, is that, in an arbitrary commutative precartesian abstract Kleisli category **K** in which every map is copyable, there is a conflict as to the correct identification of a subcategory of "total" maps. In order to obtain **K** as a Kleisli category over its category of "total" maps it is necessary to take the thunkable maps as the "total" maps. However, in order for the "total" maps to agree with the notion of totality determined by the p-category structure, it is necessary to take the p-total (i.e. discardable) maps as the "total" maps. In general every thunkable map is discardable, but not vice versa. In order to be able to match up the partiality and Kleisli structure it is necessary for the converse to hold too. It is this property that fails in examples such as the Kleisli category of the $(-)^2$ monad. It is achieved by adopting a stronger notion of abstract Kleisli p-category.

Definition 26 An *abstract Kleisli p-category* is given by a commutative precartesian abstract Kleisli category in which the diagram below commutes:



(10)

Proposition 27 *Let L be a commutative strong monad on $\mathbf{C}$. Then L is an equational lifting monad if and only if $\mathbf{C}_L$ (construed as a commutative precartesian abstract Kleisli category) is an abstract Kleisli p-category.*

Proof. By the definitions of ϑ, pairing, tensor, and composition of $\mathbf{C}_L$, the two sides of diagram (10) are the two sides of diagram (5).

By the above correspondence and the relevance property of equational lifting monads (Proposition 6), we do indeed have that every morphism in an abstract Kleisli p-category is copyable. (We give an alternative direct proof in Proposition 30 below.) Thus an abstract Kleisli p-category is indeed a p-category. In fact, as we shall see, equation (10) is equivalent to saying that every morphism is *strongly copyable* in the sense of the following definition:

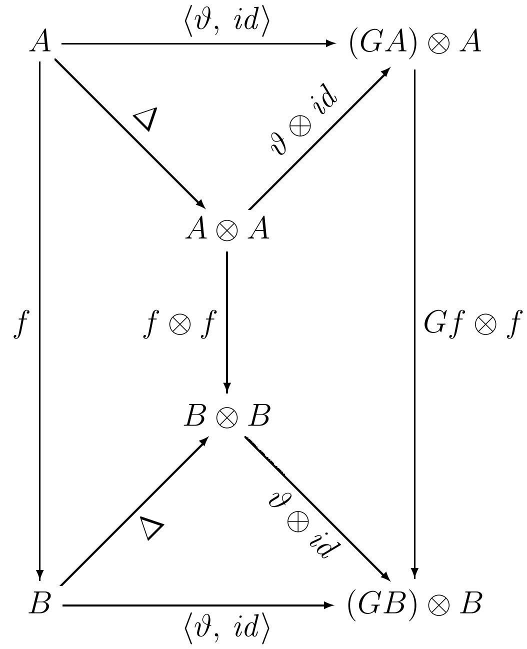
Definition 28 A morphism $A \xrightarrow{f} B$ of a commutative precartesian abstract Kleisli category is called *strongly copyable* if

$$\begin{array}{ccc} A & \xrightarrow{\langle \vartheta,\, id \rangle} & (GA) \otimes A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (Gf) \otimes f} \\ B & \xrightarrow{\langle \vartheta,\, id \rangle} & (GB) \otimes B \end{array} \qquad (11)$$

Incidentally, copyable and strongly copyable morphisms can be defined for arbitrary (even non-commutative) precartesian abstract Kleisli categories. The strongly copyable morphisms always form a premonoidal subcategory, whereas the copyable morphisms are not generally closed under composition in the non-commutative case (see [6]).

Lemma 29 *In any commutative precartesian abstract Kleisli category, every thunkable morphism is strongly copyable, and every strongly copyable morphism is copyable.*

Proof. For the first claim, let $A \xrightarrow{f} B$ be thunkable (and hence copyable), and consider



The second claim holds by simply appending $\varepsilon_B \otimes id_B$ to the two legs of diagram (11).

Proposition 30 *For a commutative precartesian abstract Kleisli category* $\mathbf{K}$*, the following are equivalent:*

(1) $\mathbf{K}$ *is an abstract Kleisli p-category.*
(2) Every morphism of $\mathbf{K}$ *is strongly copyable.*
(3) Every morphism of $\mathbf{K}$ *is copyable, and for each object* A*, the morphisms* $GA \xrightarrow{\varepsilon} A$ *and* $GA \xrightarrow{!} I$ *are jointly monic.*
(4) For all objects A *and* B*,* $A \otimes B \xrightarrow{\pi_1} A$ *and* $A \otimes B \xrightarrow{\pi_2} B$ *are jointly monic, and* $\overline{\varepsilon_A} = \vartheta_A \circ \varepsilon_A$ *(where* $\overline{\varepsilon_A}$ *is the domain map).*

Proof. For 1⇔2, first observe that diagram (10) is immediately equivalent to all components of ε being strongly copyable. To prove that this implies all morphisms are strongly copyable, note that the strongly copyable maps form a subcategory of $\mathbf{K}$ containing all thunkable maps. Now for any $A \xrightarrow{f} B$, we have $f = f \circ \varepsilon \circ \vartheta = \varepsilon \circ Gf \circ \vartheta$. But then ε strongly copyable implies f strongly copyable, because Gf and ϑ are thunkable.

For 2⇒3, suppose that every morphism is strongly copyable. By Lemma 29 is remains to prove that ε_A and $!_{LA}$ are jointly monic. Suppose that $f, g : A \longrightarrow GB$ such that $\varepsilon \circ f = \varepsilon \circ g$ and $! \circ f = ! \circ g$. Because $[-]$ is natural, we get $G\varepsilon \circ [f] = [\varepsilon \circ f] = [\varepsilon \circ g] = G\varepsilon \circ [f]$, and analogously, $G! \circ [f] = G! \circ [g]$. Because all morphisms in the images of $[-]$ and G are thunkable, they are copyable, and therefore $\langle G\varepsilon, G!\rangle \circ [f] = \langle G\varepsilon, G!\rangle \circ [g]$. The morphism $\langle G\varepsilon, G!\rangle$ is a split epi because

$$\begin{aligned}
[\pi_1 \circ (id \otimes \varepsilon)] \circ \langle G\varepsilon, G!\rangle &= [\pi_1 \circ (id \otimes \varepsilon) \circ \langle G\varepsilon, G!\rangle] \quad \text{(naturality of } [-])\\
&= [\pi_1 \circ ((G\varepsilon)\otimes !) \circ \langle id, \varepsilon\rangle] \quad \text{(naturality of } \varepsilon)\\
&= [G\varepsilon \circ \pi_1 \circ \langle id, \varepsilon\rangle] \quad \text{(finite products on } \mathbf{K}_t)\\
&= [G\varepsilon \circ \pi_1 \circ \langle \vartheta, id\rangle \circ \varepsilon] \quad \text{(diagram (10))}\\
&= [G\varepsilon \circ \vartheta \circ \varepsilon] \quad \text{(finite products on } \mathbf{K}_t)\\
&= [[\varepsilon] \circ \varepsilon]\\
&= id \quad \text{(because } [\varepsilon] = id)
\end{aligned}$$

Therefore we have $[f] = [g]$, which implies $f = g$.

Now for 3⇒4. To see that π_1 and π_2 are jointly monic, suppose that $\pi_1 \circ f = \pi_1 \circ g$ and $\pi_2 \circ f = \pi_2 \circ g$. We have $f = \langle \pi_1, \pi_2\rangle \circ f$, which by copyability of f is equal to $\langle \pi_1 \circ f, \pi_2 \circ f\rangle$. This is equal to $\langle \pi_1 \circ g, \pi_2 \circ g\rangle$, which is equal to g. To prove $\overline{\varepsilon_A} = \vartheta_A \circ \varepsilon_A$ we use that ε and ! are jointly monic. We have $\varepsilon \circ \vartheta \circ \varepsilon = \varepsilon$, which by Proposition 24(1) is equal to $\varepsilon \circ \overline{\varepsilon}$. By definition we have $! \circ \overline{\varepsilon} = ! \circ \pi_1 \circ \langle id, \varepsilon\rangle$, which because of the finite products on $\mathbf{K}_t$ is equal to $! \circ \pi_2 \circ \langle id, \varepsilon\rangle$. This, by Lemma 20, is equal to $! \circ \varepsilon$, which because of the finite products on $\mathbf{K}_t$ is equal to $! \circ \vartheta \circ \varepsilon$.

To prove 4⇒1. We prove diagram (10) with the joint monicity of π_1 and π_2. We have

$$\begin{aligned}
\pi_1 \circ \langle id, \varepsilon\rangle &= \overline{\varepsilon} = \vartheta \circ \varepsilon = \pi_1 \circ \langle \vartheta, id\rangle \circ \varepsilon\\
\pi_2 \circ \langle id, \varepsilon\rangle &= \varepsilon = \pi_2 \circ \langle \vartheta, id\rangle \circ \varepsilon
\end{aligned}$$

We end this section by showing that in abstract Kleisli p-categories the thunkable and discardable (i.e. p-total) maps coincide. Thus the definition of abstract Kleisli p-category overcomes the possible conflict, discussed earlier, between the two possible notions of totality.

Proposition 31 *In any abstract Kleisli p-category a morphism is discardable if and only if it is thunkable.*

Proof. Only the implication discardable implies thunkable is in question. Accordingly, suppose $A \xrightarrow{f} B$ is discardable. We must show that $\vartheta \circ f = Gf \circ \vartheta : A \longrightarrow GB$. We use the joint monicity of $!_{GB}$ and ε_B. On the one hand $!_{GB} \circ Gf \circ \vartheta = !_A = !_{GB} \circ \vartheta \circ f$ because all components are discardable. On the other hand $\varepsilon_B \circ Gf \circ \vartheta = f = \varepsilon_B \circ \vartheta \circ f$, easily.

6 Dominical abstract Kleisli p-categories

The notion of abstract Kleisli p-category precisely captures the Kleisli categories of equational lifting monads. In this section we characterise the Kleisli categories of dominical lifting monads, and show that any abstract Kleisli p-category fully embeds in such a *dominical* abstract Kleisli p-category.

Definition 32 We say that a commutative precartesian abstract Kleisli category is a *dominical abstract Kleisli p-category* if there exists a dominion $\mathbf{D}$ on $\mathbf{K}_t$ such that $J : \mathbf{K}_t \to \mathbf{K}$ is isomorphic to the inclusion functor $J' : \mathbf{K}_t \to \mathbf{Ptl}_{\mathbf{D}}(\mathbf{K}_t)$ (via an isomorphism of categories $\mathbf{K} \cong \mathbf{Ptl}_{\mathbf{D}}(\mathbf{K}_t)$).

Proposition 33 *Let L be a commutative strong monad on $\mathbf{C}$. Then L is an dominical lifting monad if and only if L satisfies the equaliser requirement and $\mathbf{C}_L$ (construed as a commutative precartesian abstract Kleisli category) is a dominical abstract Kleisli p-category.*

Thus any dominical abstract Kleisli p-category is indeed an abstract Kleisli p-category. The next result characterises when the converse holds. In form, it is very similar to [16, Theorem 1.7]. However, the proof will demonstrate the importance of the coincidence of the different notions of totality discussed in the previous section.

Recall that an idempotent $A \xrightarrow{a} A$ in a category $\mathbf{K}$ is said to *split* if there exist maps $A \xrightarrow{r} A' \xrightarrow{m} A$ such that $r \circ m = id_{A'}$ and $m \circ r = a$. By Corollary 25, every domain map $A \xrightarrow{\overline{f}} A$ in an abstract Kleisli p-category is idempotent.

Theorem 34 *An abstract Kleisli p-category is dominical if and only if every domain map $A \xrightarrow{\overline{f}} A$ splits.*

We shall only sketch the proof of Theorem 34, as most of the details can be found in, e.g., [16,3]. The necessity of domain maps splitting is easily shown (see, e.g., [16]). Thus we concentrate on the sufficiency.

It is instructive to work at the generality of an arbitrary category $\mathbf{K}$ with an operation mapping every $A \xrightarrow{f} B$ to an endomorphism $A \xrightarrow{\overline{f}} A$ satisfying the four equalities of Proposition 24. A category with such structure is called a *restriction category* in [3]. In any restriction category, the definition of *p-totality* (Definition 22) makes sense, and we write $\mathbf{K}_{pt}$ for the collection of p-total maps in $\mathbf{K}$, which is easily shown to form a subcategory of $\mathbf{K}$. Note also that the properties of Corollary 25 are satisfied by $\mathbf{K}$.

Assume now that every domain map splits in $\mathbf{D}$. Following [3, Section 3.3], we identify a dominion $\mathbf{D}$ on $\mathbf{K}_{pt}$ such that the inclusion $\mathbf{K}_{pt} \to \mathbf{Ptl_D}(\mathbf{K}_{pt})$ is isomorphic to the inclusion $\mathbf{K}_{pt} \to \mathbf{K}$ (via an isomorphism $\mathbf{K} \cong \mathbf{Ptl_D}(\mathbf{K}_{pt})$). The dominion is defined by:

$$\mathbf{D} \;=\; \{A' \xrightarrow{m} A \mid \text{some domain map } \overline{f} \text{ splits as } A \longrightarrow A' \xrightarrow{m} A\}$$

Note that each such m is mono and hence, by Corollary 25, in $\mathbf{K}_{pt}$. The proof that this forms a dominance uses only properties derivable from the equalities of Proposition 24. Here we give only the construction of the required pullbacks. Suppose then that $A \xrightarrow{f} B$ is in $\mathbf{K}_{pt}$ and $B' \xrightarrow{m} B$ is in $\mathbf{D}$. Thus $B' \xrightarrow{m} B$ is obtained by splitting some domain map $B \xrightarrow{e} B$ as $B \xrightarrow{r} B' \xrightarrow{m} B$. Let $A \longrightarrow A' \xrightarrow{m'} A$ be obtained by splitting $\overline{f \circ e}$. Then the square below is the required pullback of m along f (and, by definition, its left edge is indeed in $\mathbf{D}$).

$$\begin{array}{ccc} A' & \xrightarrow{\;r \circ f \circ m'\;} & B' \\ {\scriptstyle m'}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ A & \xrightarrow{\;\;f\;\;} & B \end{array}$$

The proof that this is indeed a pullback diagram, along with the other details required to show that $\mathbf{D}$ is a dominion, can be found in [3, Section 3.3].

We now show such that $\mathbf{K} \cong \mathbf{Ptl_D}(\mathbf{K}_{pt})$. Given a partial map $A \overset{m}{\hookleftarrow} A' \xrightarrow{f} B$ in $\mathbf{Ptl_D}(\mathbf{K}_{pt})$, let $A \xrightarrow{r} A' \xrightarrow{m} A$ be any splitting witnessing that m is in $\mathbf{D}$, then the associated map in $\mathbf{K}$ is given by $A \xrightarrow{f \circ r} B$ (it can be shown that $f \circ r$ is independent of the choice of splitting). Conversely, given any map $A \xrightarrow{f} B$ in $\mathbf{K}$, let $A \longrightarrow A' \xrightarrow{m} A$ split $\overline{f}$. Then the associated partial map in $\mathbf{Ptl_D}(\mathbf{K}_{pt})$ is given by $A \overset{m}{\hookleftarrow} A' \xrightarrow{f \circ m} B$ (it does indeed hold that $f \circ m$ is in $\mathbf{K}_{pt}$). That these constructions are mutually inverse can be verified from the properties of Proposition 24 and Corollary 25. This isomorphism is easily shown to commute with the inclusion functors $\mathbf{K}_{pt} \to \mathbf{Ptl_D}(\mathbf{K}_{pt})$ and $\mathbf{K}_{pt} \to \mathbf{K}$.

The proof of Theorem 34 is now easily wrapped up. Assume that $\mathbf{K}$ is an abstract Kleisli p-category. By Propositions 23 and 31 the categories $\mathbf{K}_t$ and $\mathbf{K}_{pt}$ coincide. Therefore, the facts established above for $\mathbf{K}_{pt} \to \mathbf{K}$ apply also to $\mathbf{K}_t \to \mathbf{K}$, hence $\mathbf{K}$ is indeed a dominical abstract Kleisli p-category. (We stress again that the coincidence of $\mathbf{K}_t$ and $\mathbf{K}_{pt}$ is vital to this argument.)

In the remainder of this section, we use Theorem 34 to show that every abstract Kleisli p-category has a full and faithful structure-preserving embedding into a dominical abstract Kleisli p-category. Given Theorem 34 and the analogous constructions in, e.g., [16,3], it is no surprise that we obtain this embedding via a formal idempotent splitting. However, the presence of abstract Kleisli category structure significantly complicates the verification of the construction.

Definition 35 For any category $\mathbf{K}$, and any collection S of idempotents of $\mathbf{K}$, the category $\mathbf{Split}_S(\mathbf{K})$ has:

Objects — idempotents $A \xrightarrow{a} A$ in S

Morphisms — the morphisms from $A \xrightarrow{a} A$ to $B \xrightarrow{b} B$ are those $A \xrightarrow{f} B$ such that $b \circ f = f = f \circ a$. The identity on an object a is given by a itself. Composition is inherited from $\mathbf{K}$.

Observe that if S contains all identities in $\mathbf{K}$ then there is a full and faithful functor $I : \mathbf{K} \to \mathbf{Split}_S(\mathbf{K})$ mapping each object A to id_A. Moreover, for every idempotent $a \in S$ the idempotent $I(a)$ splits in $\mathbf{Split}_S(\mathbf{K})$. Indeed, $\mathbf{Split}_S(\mathbf{K})$ is obtained from $\mathbf{K}$ be freely adding splittings of idempotents in S.

It is common experience in category theory that, under suitable conditions on S, the category $\mathbf{Split}_S(\mathbf{K})$ has all the structure of $\mathbf{K}$ and the functor I is structure preserving. In our case, we wish to split idempotents in an abstract Kleisli p-category and retain this structure. The proposition below provides the appropriate conditions on S for this to be possible.

Proposition 36 *Suppose that* $\mathbf{K}$ *is an abstract Kleisli p-category, and* S *is a collection of idempotents of* $\mathbf{K}$ *that contains all identities and is closed under the application of* G *and* $\otimes$*. Then* $\mathbf{Split}_S(\mathbf{K})$ *is also an abstract Kleisli p-category, and* $I : \mathbf{K} \to \mathbf{Split}_S(\mathbf{K})$ *strictly preserves all the commutative precartesian abstract Kleisli category structure.*

The following Lemma, which follows directly from the joint monicity of ε_A and $!_{GA}$, plays a key rôle in the proof of Proposition 36:

Lemma 37 *If* $\mathbf{K}$ *is an abstract Kleisli p-category then, for every idempotent* $A \xrightarrow{a} A$*, it holds that* $\vartheta \circ a = Ga \circ \vartheta \circ a : A \longrightarrow GA$*.*

Proof of Proposition 36. We use Proposition 19. First we must show that $\mathbf{Split}_S(\mathbf{K})$ forms an abstract Kleisli category. The required structure is defined as follows:

$$\begin{aligned}
G'a &= Ga \quad \text{(object part of } G'\text{)} \\
G'f &= Gf \quad \text{(morphism part of } G'\text{)} \\
\vartheta'_a &= Ga \circ \vartheta \circ a \\
\varepsilon'_a &= a \circ \varepsilon \circ Ga
\end{aligned}$$

Checking the equations that are need for an abstract Kleisli category is straightforward with Lemma 37. For the precartesian structure, define

$$\begin{aligned}
a \otimes' b &= a \otimes b \quad \text{(object part of } \otimes\text{)} \\
f \otimes' g &= f \otimes g \quad \text{(morphism part of } \otimes\text{)} \\
I' &= id_I \\
\Delta'_a &= (a \otimes a) \circ \Delta \circ a \\
a \otimes' b \xrightarrow{\pi'_1} a &= a \circ \pi_1 \circ (a \otimes b) \\
a \otimes' b \xrightarrow{\pi'_2} b &= b \circ \pi_2 \circ (a \otimes b) \\
!'_a &= ! \circ a
\end{aligned}$$

That $\otimes'$ is a functor is obvious. So it remains to prove the four conditions of Proposition 19.

For Condition 1, let $f \in (\mathbf{Split}_S(\mathbf{K}))(a, b)$. By Proposition 30, every morphism of $\mathbf{K}$ is copyable. Therefore every morphism $g \in (\mathbf{Split}_S(\mathbf{K}))(a, b)$ is copyable, because (we write bb for $b \otimes b$ etc.):

$$\Delta' \circ g = bb \circ \Delta \circ b \circ g = bb \circ \Delta \circ g \circ a = bb \circ gg \circ \Delta \circ a =$$
$$gg \circ aa \circ \Delta \circ a = gg \circ \Delta'$$

To see that $[f]'$ is discardable, note that by Lemma 37 we have

$$[f]' = G'f \circ \vartheta' = Gf \circ Ga \circ \vartheta \circ a = Gf \circ \vartheta \circ a = [f] \circ a \tag{12}$$

Therefore we have

$$\begin{aligned}
!' \circ [f]' &= ! \circ [Ga \circ f] \circ a \\
&= ! \circ a \quad \text{(discardability in } \mathbf{K}_t\text{)} \\
&= !'
\end{aligned}$$

Now for condition 2. For each idempotent $a \in S$, the morphism Δ'_a is thunkable because

$$
\begin{aligned}
\vartheta' \circ \Delta' &= G(aa) \circ \vartheta \circ aa \circ aa \circ \Delta \circ a \\
&= G(aa) \circ \vartheta \circ \Delta \circ a \quad \text{(because every morphism is copyable)} \\
&= G(aa) \circ G\Delta \circ \vartheta \circ a \quad \text{(because } \Delta \text{ is thunkable)} \\
&= G(aa) \circ G\Delta \circ Ga \circ Ga \circ \vartheta \circ a \quad \text{(by Lemma 37)} \\
&= G'\Delta' \circ \vartheta'
\end{aligned}
$$

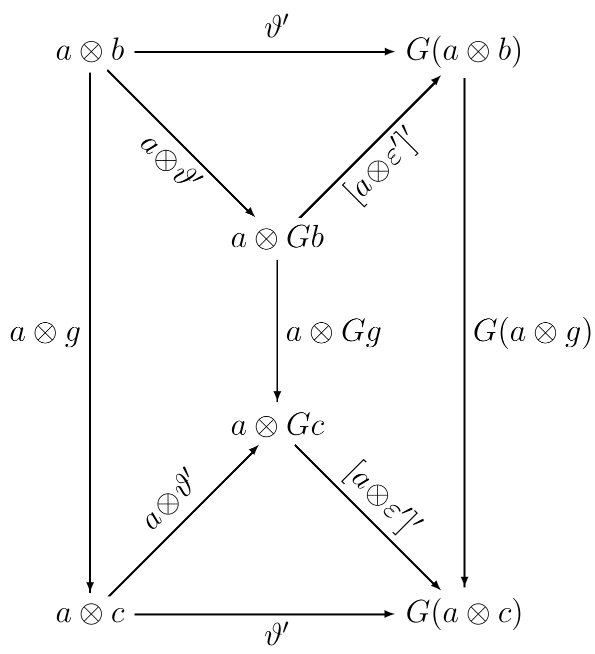
The morphism $!'_a$ is thunkable because

$$
\begin{aligned}
\vartheta' \circ !' &= Gid_I \circ \vartheta \circ id_I \circ ! \circ a \\
&= \vartheta \circ ! \circ a \\
&= G! \circ \vartheta \circ a \quad \text{(because ! is thunkable)} \\
&= G! \circ Ga \circ Ga \circ \vartheta \circ a \quad \text{(by Lemma 37)} \\
&= G!' \circ \vartheta'
\end{aligned}
$$

The morphism $\pi'_1 : ab \to a$ is thunkable because

$$
\begin{aligned}
\vartheta' \circ \pi'_1 &= Ga \circ \vartheta \circ a \circ a \circ \pi_1 \circ ab \\
&= Ga \circ \vartheta \circ \pi_1 \circ ab \quad \text{(by Lemma 20)} \\
&= Ga \circ G\pi_1 \circ \vartheta \circ ab \quad \text{(because } \pi_1 \text{ is thunkable)} \\
&= Ga \circ G\pi_1 \circ G(ab) \circ G(ab) \circ \vartheta \circ ab \quad \text{(by Lemma 37)} \\
&= G\pi'_1 \circ \vartheta'
\end{aligned}
$$

A similar argument works for π'_2. To prove that all morphisms $a \otimes' [f]'$ are thunkable, we prove the stronger claim that the thunkable morphisms are closed under $a \otimes (-)$. To see this, let $g \in (\mathbf{Split}_S(\mathbf{K}))_t(b, c)$, and consider



The left square commutes because g is thunkable. The upper triangle commutes because $\mathbf{Split}_S(\mathbf{K})$ is an abstract Kleisli category:

$$\begin{aligned}[a \otimes \varepsilon']' \circ a \otimes \vartheta' &= [id_a \otimes \varepsilon']' \circ id_a \otimes \vartheta' \\ &= [(id_a \otimes \varepsilon') \circ (id_a \otimes \vartheta')]' \quad \text{(because } [-]' \text{ is natural)} \\ &= [id_a \otimes id_b]' = \vartheta'\end{aligned}$$

The lower triangle commutes for similar reasons. For the right square, consider

$$\begin{aligned}[a \otimes \varepsilon']' \circ (a \otimes Gg) &= [id_a \otimes \varepsilon']' \circ (id_a \otimes Gg) \\ &= [(id_a \otimes \varepsilon') \circ (id_a \otimes Gg)]' \\ &= [(id_a \otimes g) \circ (id_a \otimes \varepsilon')]' \\ &= G(id_a \otimes g) \circ [id_a \otimes \varepsilon]' \\ &= G(a \otimes g) \circ [a \otimes \varepsilon]'\end{aligned}$$

So diagram (6) commutes, and hence the thunkable morphisms of $\mathbf{Split}_S(\mathbf{K})$ are closed under $a \otimes (-)$ for each object a. By a symmetric argument, they are closed under $(-) \otimes a$ too.

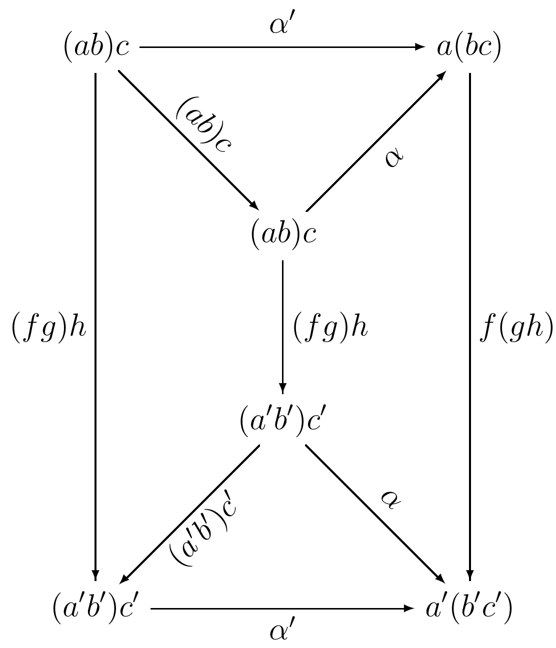
Now for Condition 3. Because every morphism of $\mathbf{K}$ is copyable, we have

$$\langle f, g\rangle' = fg \circ \Delta' = fg \circ aa \circ \Delta \circ a = fg \circ \Delta \circ a = \langle f, g\rangle \circ a \qquad (13)$$

For the naturality of the associativity map, define

$$\begin{aligned}\alpha &= \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2\rangle\rangle \\ \alpha' &= \langle \pi_1' \circ \pi_1', \langle \pi_2' \circ \pi_1', \pi_2'\rangle\rangle'\end{aligned}$$

We prove that the following diagram commutes:



The left square and the right square obviously commute. The upper triangle commutes because

$$\begin{aligned}
\alpha' &= \langle \pi_1' \circ \pi_1',\ \langle \pi_2' \circ \pi_1',\ \pi_2' \rangle\rangle' \\
&= \langle a \circ \pi_1 \circ ab \circ \pi_1 \circ (ab)c,\ \langle b \circ \pi_2 \circ ab \circ \pi_1 \circ (ab)c,\ c \circ \pi_2 \circ (ab)c\rangle\rangle \circ (ab)c \\
&= \langle \pi_1 \circ \pi_1 \circ (ab)c,\ \langle \pi_2 \circ \pi_1 \circ (ab)c,\ \pi_2 \circ (ab)c\rangle\rangle \circ (ab)c \quad \text{(by Lemma 20)} \\
&= \langle \pi_1 \circ \pi_1,\ \langle \pi_2 \circ \pi_1,\ \pi_2\rangle\rangle \circ (ab)c \quad \text{(because } (ab)c \text{ is copyable in } \mathbf{K}) \\
&= \alpha \circ (ab)c
\end{aligned}$$

The lower square commutes for similar reasons. Proving the naturality of the remaining structural isos is quite similar and a bit easier, so we leave it as an exercise.

Checking Condition 4 is straightforward.

To see that $\mathbf{Split}_S(\mathbf{K})$ is an abstract Kleisli p-category, it suffices to verify diagram (10), but this follows easily from (10) in $\mathbf{K}$.

It remains to verify that the functor $I : \mathbf{K} \to \mathbf{Split}_S(\mathbf{K})$ strictly preserves all the commutative precartesian abstract Kleisli p-category structure. But this is immediate from the definition of the structure on $\mathbf{Split}_S(\mathbf{K})$. This completes the proof of Proposition 36.

One intriguing fact about the above proof is that all the structure of an abstract Kleisli p-category is used even to show that $\mathbf{Split}_S(\mathbf{K})$ is a (commutative precartesian) abstract Kleisli category (that it is also an abstract Kleisli p-category follows easily). Thus it appears that, in general, idempotent splitting does not preserve (commutative precartesian) abstract Kleisli category structure. Abstract Kleisli p-categories are very much a special case.

Corollary 38 *Suppose* $\mathbf{K}$ *is an abstract Kleisli p-category. Let* S *be the collection of all idempotents in* S*. Then* $\mathbf{Split}_S(\mathbf{K})$ *is a dominical abstract Kleisli p-category, and* $I : \mathbf{K} \to \mathbf{Split}_S(\mathbf{K})$ *is a full and faithful functor strictly preserving all structure.*

Proof. The collection of all idempotents in $\mathbf{K}$ trivially satisfies the conditions for Proposition 36 to apply. Thus $\mathbf{Split}_S(\mathbf{K})$ is an abstract Kleisli p-category, and $I : \mathbf{K} \to \mathbf{Split}_S(\mathbf{K})$ is a full and faithful structure-preserving functor. That $\mathbf{Split}_S(\mathbf{K})$ is a dominical abstract Kleisli p-category follows from Theorem 34, as all idempotents in $\mathbf{Split}_S(\mathbf{K})$ split (not just the domain maps).

We end the section with a remark about the above proof. In order to expand an abstract Kleisli p-category $\mathbf{K}$ to a dominical abstract Kleisli p-category, it is not really necessary to split *all* idempotents in $\mathbf{K}$. However, in spite of Theorem 34, it does not suffice to split only the domain maps, [5] as these do not satisfy the closure conditions required by Proposition 36. What is possible is to define S to be the least collection of maps containing all identities and closed under G, $\otimes$ and the following rule: if $a \in S$, and $f \circ a = f$, then $\overline{f} \circ a \in S$. (It follows that all maps in S are idempotents.) The dominical abstract Kleisli p-category obtained as $\mathbf{Split}_S(\mathbf{K})$ has a universal property as the free dominical abstract Kleisli p-category over $\mathbf{K}$. This universal property can be expressed in terms of a 2-adjunction between the 2-category of abstract Kleisli p-categories and its dominical subcategory, along the lines of the 2-adjunctions exhibited in [3].

7 Equational lifting monads revisited

Having taken a thorough look at the abstract Kleisli category account of lifting, we now return to our intitial viewpoint, in which the category with monad is taken as primitive. The results we have established for abstract Kleisli p-categories will be very helpful for establishing properties of equational lifting monads. Having now assembled all the pieces, we begin with a swift proof of Theorem 10.

[5] This fact was overlooked in [16, p. 127], where the claim that partial exponentiation functors in $\mathbf{C}$ extend to all of $\mathfrak{D}$-Ptl($\mathbf{D}$) appears to be incorrect.

Proof of Theorem 10. Let L be an equational lifting monad on $\mathbf{C}$. By Proposition 27, the Kleisli category $\mathbf{C}_L$ is an abstract Kleisli p-category. Moreover, as in Section 4, there is a strong monad preserving functor $U : \mathbf{C} \to (\mathbf{C}_L)_t$ (the thunkable subcategory) that is (trivially) Kleisli full and Kleisli faithful (there is an isomorphism of Kleisli categories).

Let S be the collection of all idempotents in $\mathbf{C}_L$. By Proposition 36, $\mathbf{Split}_S(\mathbf{C}_L)$ is a dominical abstract Kleisli p-category, and there is a full and faithful structure-preserving functor $I : \mathbf{C}_L \to \mathbf{Split}_S(\mathbf{C}_L)$. The functor I restricts to a strong monad preserving functor $I_t : (\mathbf{C}_L)_t \to (\mathbf{Split}_S(\mathbf{C}_L))_t$ which is (Kleisli) full and (Kleisli) faithful. Moreover, by Proposition 33, the induced monad L' on $(\mathbf{Split}_S(\mathbf{C}_L))_t$ is dominical.

Thus $I_tU : \mathbf{C} \to (\mathbf{Split}_S(\mathbf{C}_L))_t$ is a Kleisli full and Kleisli faithful dominical representation of L, completing the proof of Theorem 10.

Let us consider to what extent the extensive detour through abstract Kleisli p-categories was actually helpful in the proof of Theorem 10 (of course it was not strictly *necessary* as all results for abstract Kleisli categories can be equivalently formulated as results about monads). In order to prove Theorem 10, it is necessary to construct a Kleisli full and Kleisli faithful dominical representation of an equational lifting monad L on $\mathbf{C}$. What is required to achieve this is best appreciated by understanding the additional properties that an equational lifting monad needs to satisfy in order to be dominical. There is in fact an immediate characterisation available from Theorem 34.

Theorem 39 *A strong monad L is a dominical lifting monad if and only if it is an equational lifting monad satisfying the equaliser requirement, and all domain maps split in the Kleisli category $\mathbf{C}_L$.*

One can view the above proof of Theorem 10 as dividing into two natural stages. The first, the construction of $(\mathbf{C}_L)_t$ provides a representation of $(\mathbf{C}, L)$ into $((\mathbf{C}_L)_t, L')$ satisfying the equaliser requirement. This is Kleisli full and Kleisli faithful because $(\mathbf{C}_L)_t$ is defined as the thunkable maps within the original Kleisli category. This construction already makes use of abstract Kleisli categories, applying the reflection theorem of [6] to equational lifting monads. However, it would be quite feasible to reformulate this part of the proof entirely in terms of the equational lifting monad structure, by verifying directly that $(\mathbf{C}_L)_t$ carries a suitable equational lifting monad (although this is not entirely trivial as one must verify, e.g., that $(\mathbf{C}_L)_t$ has products)

The second stage of the proof provides the required idempotent splittings. For the purpose of proving Theorem 10, one needs to know that if one splits suitable idempotents in $\mathbf{C}_L$ then the resulting category is still the Kleisli category of an (appropriate) equational lifting monad. In order to verify such a property, it is indispensable to have a direct axiomatization of such Kleisli categories so that the structure that must be preserved under by idempotent spliting is identified. The notion of abstract Kleisli p-category provides such an axiomatization.

We end this section with some curiosities. Theorem 39 characterises dominical lifting monads as those equational lifting monads satisfying the equaliser requirement such that, in addition, every "partial map" in the Kleisli category $\mathbf{C}_L$ has a "domain" mono in $\mathbf{C}$, where the "domain" mono is obtained via an idempotent splitting in $\mathbf{C}_L$. As the induced $\mathbf{D}$ is a dominion, this condition guarantees that, for every morphism $A \xrightarrow{g} LB$ in $\mathbf{C}$, there exists a pullback

$$\begin{array}{ccc} A' & \xrightarrow{f} & B \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta} \\ A & \xrightarrow{g} & LB \end{array} \tag{14}$$

in $\mathbf{C}$. What is perhaps surprising is that the existence of such pullbacks alone (which seems a plausible definition of "domain" mono in $\mathbf{C}$) is not itself sufficient for an equational lifting monad satisfying the equaliser requirement to be dominical (it does not alone guarantee that the domain maps split in $\mathbf{C}_L$). There is an amusing counterexample in the category **Ass** of assemblies over $\mathbb{N}$. The functor $L_0 : \mathbf{Ass} \to \mathbf{Ass}$ defined in [14, Theorem 5.4(v)] carries the structure of an equational lifting monad satisfying the equaliser requirement. However, although the above pullbacks exist (indeed **Ass** is finitely complete), this monad is not dominical. (Interestingly, the extension of L_0 to a functor $L : \mathbf{ModAss} \to \mathbf{ModAss}$ on *modified assemblies*, see [14], does yield a dominical lifting monad, providing a full and failthful dominical representation of L_0.)

Finally, we prove a curious fact whose importance has been impressed upon us by Paul Taylor.

Proposition 40 *For an equational lifting monad* $(\mathbf{C}, L)$*, the following are equivalent.*

(1) L satisfies the equaliser requirement.
(2) The functor $J : \mathbf{C} \to \mathbf{C}_L$ is comonadic.

Proof. We work with the abstract Kleisli p-category structure on $\mathbf{C}_L$. There is an evident full and faithful functor $(\mathbf{C}_L)_t \to G\text{-Coalg}(\mathbf{C}_L)$, which maps an object A to the comonad coalgebra $A \xrightarrow{\vartheta} GA$ in $\mathbf{C}_L$. The comparison functor is given by the composite $\mathbf{C} \to (\mathbf{C}_L)_t \to G\text{-Coalg}(\mathbf{C}_L)$. We must show that this is an isomorphism if and only if the equaliser requirement holds. But, as remarked in Section 4, the functor $\mathbf{C} \to (\mathbf{C}_L)_t$ is an isomorphism if and only if the equaliser requirement holds. Thus it suffices to show that the functor $(\mathbf{C}_L)_t \to G$-Coalg$(\mathbf{C}_L)$ is always an isomorphism. Accordingly, let $A \xrightarrow{a} GA$ be any comonad coalgebra for G. We show that $a = \vartheta_A$. As a is a comonad coalgebra, $\varepsilon \circ a = id_A$, thus a is mono and hence, by Corollary 25 and Proposition 31, thunkable. But then $a = [\varepsilon \circ a]$ so $a = [id_A] = \vartheta_A$ as required.

8 Discussion

The work presented in this paper grew out of earlier joint work of the first author and Rosolini [2]. The motivation behind the present paper is similar to that of [2], but the approaches are very different in detail. On the one hand, the scope of [2] is more general, as they do not always assume that $\mathbf{C}$ has finite products (as we do here). However, in the case that $\mathbf{C}$ does have finite products, the characterisation of lifting monads given by Theorem 6 of [2] is closely related to our notion of equational lifting monad. The conditions given in statement (ii) there amount to L being an equational lifting monad satisfying the equaliser property in which, in addition, L preserves existing pullbacks in $\mathbf{C}$ (this claim takes some proving). In [2] it is shown that, under such conditions, $\mathbf{C}$ fully embeds in a category $\mathbf{C}_t$ with dominion such that $\mathbf{C}_L$ fully embeds in the associated category of partial maps. Moreover, an explicit construction is given of the free such $\mathbf{C}_t$. However, the major difference with our work is that the $\mathbf{C}_t$ constructed in [2] does not itself have a lifting monad (of any sort) acting on it. Thus, in the present paper, we obtain a stronger representation theorem under weaker conditions (we do not require L to preserve pullbacks — the functor $L_0 : \mathbf{Ass} \to \mathbf{Ass}$, see Section 7, is an example that does not). We also separate out precisely the equational properties of lifting monads from the non-equational properties.

At first sight, our equation (5) may seem rather curious. It is perhaps illuminating to consider the significance of the equation within Moggi's computational lambda-calculus [11]. It appears that equation (5) corresponds to the intersubstitutivity of e with x in the body M of let $x = e$ in M. It would be interesting to see if the completeness of such a formulation might lead to a simplified meta-theory for the partial λ-calculus [10]. Such an investigation would require extending the results in this paper to cover partial function spaces, which is itself a (probably straightforward) programme of independent interest.

One of the applications we have of the work in this paper is to establish properties of recursion in axiomatic domain theory. A general axiomatic analysis of recursion has been carried out in [15], establishing equational completeness assuming the existence of sufficiently many final coalgebras. In the presence of an equational lifting monad all the necessary final coalgebras can be constructed from a natural numbers object in **C** [15].

Although we have stressed the computer science motivation for the work in this paper, we believe that our equational characterisation of partial map classifiers is of independent mathematical interest. There are many other similar projects possible that might worth investigating. One question is whether there is an equational characterisation of those (subpowerset) monads whose Kleisli categories can be viewed as categories of relations. Such monads would include all lifting monads, but also powerobject monads and various other related notions of free lattice. A natural setting would be to obtain representation theorems with respect to regular categories.

Another possible direction for generalising the work in this paper is dropping the commutativity requirement on equational lifting monads. Although we did not say it explicitly, many of the calculations do not use commutativity. When commutativity is dropped, equation (5) also includes exception monads. Whether there is any potential in this observation remains to be seen.

Finally, we should mention related work by Robin Cockett and Stephen Lack. Building on their still unpublished *restriction categories* [3], they have recently extended their axiomatization to incorporate lifting monads. Their approach is somewhat different from ours. A proper comparison must await the publication of their work.

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