On the Geometry of Interaction for Classical Logic (Extended Abstract)

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Abstract

It is well-known that weakening and contraction cause naïve categorical models of the classical sequent calculus to collapse to Boolean lattices. We introduce sound and complete models that avoid this collapse by interpreting cut-reduction by a partial order between morphisms. We provide concrete examples of such models by applying the geometry-of-interaction construction to quantaloids with finite biproducts, and show how these models illuminate cut reduction in the presence of weakening and contraction. Our models make no commitment to any translation of classical logic into intuitionistic logic and distinguish non-deterministic choices of cut-elimination.

1. Introduction

It is widely-believed that the classical sequent calculus has no decent denotational semantics, let alone an algorithmic interpretation. Indeed, Cartesian closed categories with dualizing maps are equivalent to Boolean lattices [9, 7]. This algebraic observation, however, is too crude to reveal the proof-theoretic structure to be found. In fact, the key to understanding the algebraic structure of classical proofs lies in the properties of weakening and contraction in multiple-conclusion systems and the *non-deterministic* behaviour of cut-elimination. To see the point, consider the proof

$$\Lambda \quad = \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \phi, \Delta}\,\mathsf{WR} \qquad \dfrac{\begin{array}{c}\Phi_2\\ \vdots\\ \Gamma \vdash \Delta\end{array}}{\Gamma, \phi \vdash \Delta}\,\mathsf{WL}}{\Gamma, \Gamma \vdash \Delta, \Delta}\,\mathsf{Cut}}{\Gamma \vdash \Delta}\,\mathsf{CL, CR},$$

where Φ_1 and Φ_2 are arbitrary proofs of $\Gamma \vdash \Delta$. We call this the "Lafont proof", because it is a variant of an example accredited to Lafont (cf. page 151 in [15]). The sub-proof Φ_1 is weakened on the right, and the sub-proof Φ_2 is weakened on the left. Then follows a cut, where the cut-formula is the formula ϕ introduced by the weakenings. Finally, the double occurrences of Γ and Δ are removed by contractions. (Clearly, the left and right contractions are supposed to commute with each other, so we need not specify the order in which they are applied.) The proof Λ reduces to

$$\dfrac{\dfrac{\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma \vdash \Delta\end{array}}{\Gamma, \Gamma \vdash \Delta, \Delta}\,\mathsf{WL, WR}}{\Gamma \vdash \Delta}\,\mathsf{CL, CR} \qquad \text{or to} \qquad \dfrac{\dfrac{\begin{array}{c}\Phi_2\\ \vdots\\ \Gamma \vdash \Delta\end{array}}{\Gamma, \Gamma \vdash \Delta, \Delta}\,\mathsf{WL, WR}}{\Gamma \vdash \Delta}\,\mathsf{CL, CR}.$$

Clearly, the weakenings followed by the contractions are essentially nothing (cf. [15], page 152). So both Φ_1 and Φ_2 are reducts of Λ, and the denotations of Φ_1 and Φ_2 must be equal for any semantics that *admits cut-reduction* in the sense that the reduct is denotationally equal to the redex. In summary, any denotational semantics that admits cut-reduction must identify all proofs of a sequent $\Gamma \vdash \Delta$.

There are various escapes from this collapse:

- Single-conclusion systems, where the succedent Δ contains at most one formula. (This rules out Lafont's example.) An example is Gentzen's system LJ for intuitionistic logic;
- Systems without weakening and contraction; for example, linear logic. (This too rules out the Lafont proof.) As is widely-known, multiplicative *classical* linear logic can be modelled by certain *linearly distributive categories* (formerly called "weakly distributive categories", see [3]);
- "Classical natural deduction" systems [17], where proofs may be represented as terms of the $\lambda\mu\nu$-calculus [16, 18]. Such systems do not admit all cut-reductions: the call-by-name version of $\lambda\mu\nu$ admits only the reduction of Λ to Φ_2, while the call-by-value version admits only the reduction to Φ_1. Each version corresponds to a different choice of $\neg\neg$-translation.

But, with a classical perspective, these escapes are flawed: intuitionistic logic and natural deduction systems ruin the symmetry of the classical sequent calculus; intuitionistic logic and linear logic limit the range of provable judgements. (For a detailed discussion of the non-determinism of cut-reduction, see [22].)

In this article, we present a solution that does not suffer from these flaws, analyzing more carefully what it means to "admit cut-reduction". Specifically, the present authors have introduced in [5] a kind of order-enriched category called a *classical category* whose objects model types and whose morphisms model proofs of the classical sequent calculus. For cut-reduction, whenever a proof Φ can be reduced to another proof Ψ, we only require $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ (as opposed to $\mathbf{C}\lfloor\Phi\rfloor = \mathbf{C}\lfloor\Psi\rfloor$), where $\mathbf{C}\lfloor\Phi\rfloor$ and $\mathbf{C}\lfloor\Psi\rfloor$ are the morphisms of $\mathbf{C}$ denoted by Φ and Ψ, respectively. Classical categories are a special case of symmetric linearly distributive categories [3]: they have symmetric monoidal products $\otimes$ and $\oplus$ for modelling conjunction and disjunction, respectively. To model contraction and weakening on the right, every object A is endowed with a symmetric monoid $(\nabla_A : A \oplus A \longrightarrow A, []_A : 0 \longrightarrow A)$. Dually for contraction and weakening on the left.

We shall introduce a notion of theory with judgements of the form $\Phi \preccurlyeq \Psi$ where Ψ and Φ are proofs of the same sequent $\Gamma \vdash \Delta$, such that:

- Whenever Φ can be cut-reduced to Ψ, then the judgement $\Phi \preccurlyeq \Psi$ is a theorem (Theorem 3.6). (The relation $\preccurlyeq$ is slightly greater than the reflexive-transitive-compatible closure of cut-reduction. For example, we require the standard axiom expansions (a.k.a. η-laws), i.e., the laws that allow to rewrite axioms $\overline{\phi \vdash \phi}$ in terms of axioms that involve only subformulæ of ϕ);
- Classical categories are sound with respect to this notion of theory (Theorem 3.7). This means essentially that $\Phi \preccurlyeq \Psi$ implies $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ for every classical category $\mathbf{C}$;
- We also have completeness (Theorem 3.8). That is, if $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ holds for every model $\mathbf{C}\lfloor-\rfloor$ of a theory $\mathcal{T}$, then $\Phi \preccurlyeq \Psi$ is a theorem of $\mathcal{T}$.

Examples of classical categories include the category $\mathbf{Rel}_\otimes$ of sets and relations, in which both disjunction and conjunction are modelled by the set-theoretic product (Example 3.2), every Boolean lattice (Example 3.3), and the product of any two classical categories. These examples are discussed in detail in [5].

After introducing classical categories and establishing, cut-elimination relative to $\preccurlyeq$, soundness and completeness (§ 3), proved in detail by the present authors in [5], we shall devote the rest of this article to a particularly interesting class of classical categories, $\mathcal{G}(\mathbf{Q})$, that arise from applying the Geometry of Interaction (GoI) construction to a *quantaloid* $\mathbf{Q}$ with finite biproducts (§ 4). The theorem stating that categories of the form $\mathcal{G}(\mathbf{Q})$ are classical categories (Theorem 4.2) goes significantly beyond the usual GoI construction in that it makes essential use of the quantaloid's biproducts and local order. Also, it describes an interesting characterization of monoid homomorphisms and co-monoid homomorphisms, which is closely linked with the question whether a proof makes essential use of negation (Proposition 4.6).

We shall describe how models of the form $\mathcal{G}(\mathbf{Q})$ lead to a presentation of proofs as matrices (§ 4.1), which in turn have a graphical presentation (§ 4.2).

Finally, we shall use this graphical presentation to explain how proofs can "grow" during cut-reduction in the presence of weakening and contraction (§ 5).

2. Preliminaries

The version of the classical sequent calculus to which we refer is given in Table 3 (p. 8), where the arrows are to be ignored until later in this article. As in [19], we use a system of multiplicative linear logic plus rules for weakening and contraction, and so obtain a calculus which differs from Gentzen's LK [6] only in its use of the multiplicative form of the introduction rules and the absence of explicit implication. We consider implication to be derived — that is, $\phi \Rightarrow \psi = \neg\phi \vee \psi$, and we treat the rule $\top$L as a degenerate case of WL, with $\phi = \top$, and dually for $\bot$R.

2.1. Linearly distributive categories

A linearly distributive category (originally called "weakly distributive category" [3]) is a category with two symmetric monoidal products $\otimes$ and $\oplus$ and a natural transformation $\delta : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$, called *(linear) distribution*, satisfying certain equations [3]. As is well-known, the distribution is exactly what is needed to model the cut rule. Symmetric linearly distributive categories with negation have an object $A^\perp$ for every object A, and morphisms $A^\perp \otimes A \longrightarrow 0$ and $1 \longrightarrow A \oplus A^\perp$ satisfying certain coherence conditions. They are equivalent to $*$-autonomous categories, and provide a sound and complete semantics of multiplicative linear logic [3]. A *compact closed category* is a symmetric linearly distributive category $\mathbf{C}$ with negation such that $(\mathbf{C}, \otimes, 1) = (\mathbf{C}, \oplus, 0)$.

2.2. Biproducts and quantaloids

A category $\mathbf{C}$ is said to have *finite biproducts* if, for all objects $A_1, \ldots, A_n$, there is an object $A_1 \oplus \cdots \oplus A_n$ with two families of maps

$$A_l \xrightarrow{i_l} A_1 \oplus \cdots \oplus A_n \xrightarrow{p_k} A_k \quad k, l \in \{1, \ldots, n\}$$

such that the p_k form a product cone and the i_l form a coproduct cone for $A_1, \ldots, A_n$ (cf. [10, 20]). When $n = 0$, we call the biproduct the *null object* (which is initial and terminal), and denote it by 0.

For any two objects A and B, the *zero arrow* 0_{AB} is defined to be the composite $A \longrightarrow 0 \longrightarrow B$. We write Δ_A resp. ∇_A for the evident maps $A \longrightarrow A \oplus A$ resp. $A \oplus A \longrightarrow A$. The finite biproducts induce an enrichment over the category of symmetric monoids: the operation $+$ on $\mathbf{C}(A, B)$ defined by $f + g = \nabla_B \circ (f \oplus g) \circ \Delta_A$ forms a symmetric monoid with 0_{AB} as the neutral element, and the composition of morphisms is a monoid homomorphism in both arguments (see Exercise 4(a) in § VIII.2 of [10]). Owing to the biproducts, this is the only symmetric-monoid enrichment on $\mathbf{C}$ (cf. Prop. 4 in § VIII.2 of [10]).

We are interested in the case where the hom-monoids are complete lattices. A *quantaloid* [20] is a locally small category $\mathbf{Q}$ such that every hom-set is a complete lattice, and the composition of morphisms preserves suprema in both variables. We write $f \sqcup g$ instead of $f + g$, and $\sqsubseteq$ for the corresponding order. We shall use the term "quantaloid with finite biproducts" for categories with finite biproducts that happen to be quantaloids; this is justified by the aforementioned fact that the symmetric-monoid enrichment is uniquely determined by the biproducts.

Example 2.1. The category $\mathbf{Rel}_\oplus$ (not to be confused with $\mathbf{Rel}_\otimes$ in Example 3.2), whose objects are (small) sets, and whose morphism $A \longrightarrow B$ are subsets of $A \times B$. The biproduct of sets is given by the disjoint union, and $\sqsubseteq$ turns out to be the set-theoretic inclusion.

Given objects $A_1, \cdots, A_n$ and $B_1, \cdots, B_m$, and maps $f_{lk} : A_l \longrightarrow B_k$, we write

$$\begin{pmatrix} f_{11} & \cdots & f_{n1} \\ \vdots & & \vdots \\ f_{1m} & \cdots & f_{nm} \end{pmatrix}$$

for the unique morphism $f : A_1 \oplus \cdots \oplus A_n \longrightarrow B_1 \oplus \cdots \oplus B_m$ such that $p_k \circ f \circ i_l = f_{lk}$.

For a morphism $f : A \longrightarrow A$ of a quantaloid the *reflexive-transitive closure* f^* is defined as $f^* = \bigsqcup_{i \geq 0} f^i$.

2.3. Geometry of interaction

The *Geometry of Interaction* (GoI) was introduced by Girard [12, 13, 14], in the context of modelling linear logic [11], as an attempt to provide a mathematically precise presentation of logical structure independent of any prior commitment to a choice between syntax and semantics.

More recently, Abramsky et al. [1] have presented a construction, adumbrated in lectures by Martin Hyland, which isolates the key properties of Girard's GoI. Abramsky's version in turn is a special case of a construction due to Joyal et al. [8] that sends a traced monoidal category to a tortile monoidal category.

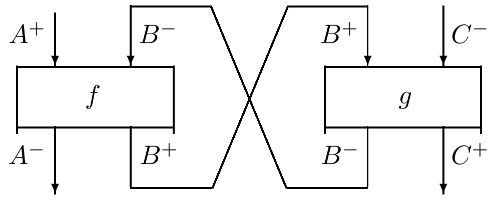


Figure 1: Symmetric feedback

The starting point of the GoI construction is a *traced symmetric monoidal category*, which is a symmetric monoidal category $\mathbf{Q}$ together with a family of functions

$$Tr^X_{A,B} : \mathbf{Q}(A \otimes X, B \otimes X) \longrightarrow \mathbf{Q}(A, B),$$

called a *trace*, which is natural in A and B, dinatural in X, and satisfies a number of conditions (see [1]).

Next, we recall the GoI construction as presented in [1].

Definition 2.2. Given a traced symmetric monoidal category $\mathbf{Q}$, the category $\mathcal{G}(\mathbf{Q})$ is defined as follows:

- Objects are pairs (A^+, A^-) of objects of $\mathbf{Q}$;
- A morphism $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ of $\mathcal{G}(\mathbf{Q})$ is a morphism $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ of $\mathbf{Q}$;
- The identity $id_{(A^+,A^-)}$ is the twist map $\sigma_{A^+A^-} : A^+ \otimes A^- \longrightarrow A^- \otimes A^+$ of $\mathbf{Q}$;
- Composition is given by *symmetric feedback*, as illustrated in Figure 1. For $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (B^+, B^-) \longrightarrow (C^+, C^-)$, the composite $g \circ f : (A^+, A^-) \longrightarrow (C^+, C^-)$ is

$$\dfrac{\dfrac{(A^+ \otimes B^-) \otimes (B^+ \otimes C^-) \xrightarrow{f \otimes g} (A^- \otimes B^+) \otimes (B^- \otimes C^+)}{(A^+ \otimes C^-) \otimes (B^- \otimes B^+) \longrightarrow (A^- \otimes C^+) \otimes (B^- \otimes B^+)}}{A^+ \otimes C^- \longrightarrow A^- \otimes C^+}\ Tr.$$

We have the following result from [1] (which essentially follows from results in [8]):

Proposition 2.3. *$\mathcal{G}(\mathbf{Q})$ is a compact closed category.*

Proof (Sketch). We have $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-)$; for $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (C^+, C^-) \longrightarrow (D^+, D^-)$, the morphism $f \otimes g : (A^+ \otimes C^+, A^- \otimes C^-) \longrightarrow (B^+ \otimes D^+, B^- \otimes D^-)$ is

$$\dfrac{(A^+ \otimes B^-) \otimes (C^+ \otimes D^-) \xrightarrow{f \otimes g} (A^- \otimes B^+) \otimes (C^- \otimes D^+)}{(A^+ \otimes C^+) \otimes (B^- \otimes D^-) \longrightarrow (A^- \otimes C^-) \otimes (B^+ \otimes D^+)}.$$

The tensor unit is $(1, 1)$ where 1 is the tensor unit of $\mathbf{Q}$. We have $(A^+, A^-)^\perp = (A^-, A^+)$. The map γ^L :

$(A^+, A^-)^\perp \otimes (A^+, A^-) \longrightarrow (1,1)$ is given by the isomorphism $(A^- \otimes A^+) \otimes 1 \longrightarrow 1 \otimes (A^+ \otimes A^-)$, and dually for $\tau^R : 1 \longrightarrow (A^+, A^-) \otimes (A^+, A^-)^\perp$. □

3. Classical categories

Next, we introduce *classical categories* as sound and complete semantics of the classical sequent calculus.

A classical category is a symmetric linearly distributive category with negation, plus some extra structure. An interpretation in a symmetric linearly distributive category $\mathbf{C}$ with negation sends a formula ϕ to an object $\mathbf{C}\lfloor\phi\rfloor$ (or $\lfloor\phi\rfloor$ in short) according to the rules

$$\lfloor\phi \wedge \psi\rfloor = \lfloor\phi\rfloor \otimes \lfloor\psi\rfloor \quad \lfloor\top\rfloor = 1$$
$$\lfloor\psi \vee \psi\rfloor = \lfloor\phi\rfloor \oplus \lfloor\psi\rfloor \quad \lfloor\bot\rfloor = 0 \quad \lfloor\neg\phi\rfloor = \lfloor\phi\rfloor^\perp.$$

(So the interpretation of formulæ is determined by the interpretation of atomic formulæ.) A proof Φ of a sequent $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$ is interpreted by a morphism

$$\mathbf{C}\lfloor\Phi\rfloor : \lfloor\phi_1\rfloor \otimes \cdots \otimes \lfloor\phi_n\rfloor \longrightarrow \lfloor\psi_1\rfloor \oplus \cdots \oplus \lfloor\psi_m\rfloor$$

For a detailed description of the interpretation, see [4].

Classical categories provide the extra structure required for interpreting weakening and contraction. To model ER, CR and WR, every object A has a *symmetric monoid* — that is, a multiplication $\nabla_A : A \oplus A \longrightarrow A$ and unit $[]_A : 0 \longrightarrow A$ satisfying the evident equations that state the associativity and symmetry of ∇_A and the neutrality of $[]_A$. We say that symmetric monoidal category *has symmetric monoids* if every object has a symmetric monoid, and three more equations hold: one that states that multiplication $\nabla_{A\oplus B}$ is definable pointwise — that is, from ∇_A and ∇_B, a similar one that states that the unit $[]_{A\oplus B}$ is definable pointwise, and the requirement $[]_0 = id$. (These equations are standard, see any of [21, 5, 4] for diagrams.) Dually, EL, CL, and WL are modelled by symmetric co-monoids, with maps $\Delta_A : A \longrightarrow A \otimes A$ and $\langle\rangle : A \longrightarrow 1$.

Definition 3.1. A *classical category* is a symmetric linearly distributive category with negation, together with a partial order $\leq_{A,B}$ on every hom-set $\mathbf{C}(A,B)$, such that:

1. The symmetric monoidal category $(\mathbf{C}, \oplus, 0)$ has symmetric monoids;
2. The symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ has symmetric co-monoids;
3. The object-indexed families of maps Δ_A, ∇_A, $\langle\rangle_A$, and $[]_A$ are lax natural transformations in the sense that for every morphism f we have

$$\Delta \circ f \leq (f \otimes f) \circ \Delta \qquad f \circ \nabla \leq \nabla \circ (f \oplus f)$$
$$\langle\rangle \circ f \leq \langle\rangle \qquad f \circ [] \leq [];$$

4. The inequalities in Table 1 hold, where the morphisms δ_2 and δ_2' are the evident maps built from the distributions δ and symmetric-monoidal isomorphisms;
5. Composition of morphisms, and the functors $\otimes$, $\oplus$ are monotonic in all arguments. (It can be shown that this implies that the negation functor too is monotonic.)

Example 3.2. $\mathbf{Rel}_\otimes$ is a (compact closed) classical category, if both $\otimes$ and $\oplus$ are defined to be evident functor that takes the set-theoretic product $\times$ (not the biproduct, which is given by the set-theoretic union). Both 1 and 0 are the singleton set $\{*\}$. Negation does nothing to objects. The map $A^\perp \otimes A \longrightarrow 1$ is $\{((x,x),*) : x \in A\}$, and dually for $0 \longrightarrow A \oplus A^\perp$. The map ∇_A is $\{((x,x),x) : x \in A\}$, and $[]_A$ is $\{(*,x) : x \in A\}$. Dually for Δ_A and $\langle\rangle_A$. The hom-order is the set-theoretic inclusion of the relations.

Example 3.3. Every Boolean lattice $\mathbf{B}$. The objects are the elements of $\mathbf{B}$, and there is at most one morphism $x \longrightarrow y$, which exists if and only if $x \leq y$. The functor $\otimes$ is the meet, and $\oplus$ is the join.

Example 3.4. If $\mathbf{C}$ and $\mathbf{C}'$ are classical categories, then so are $\mathbf{C}^{op}$ and $\mathbf{C} \times \mathbf{C}'$. Because there exist compact closed classical categories $\mathbf{C}$ with non-trivial hom-orders (e.g., $\mathbf{Rel}_\otimes$ as in the example above, or the GoI models introduced later in this article), and $\mathbf{C}'$ can be some Boolean lattice, it is evident that there are classical categories which are not compact closed and have non-trivial hom-orders.

The rule CR, with principal formula ϕ, is modelled essentially by post-composing $\nabla_{\lfloor\phi\rfloor} : \lfloor\phi\rfloor \oplus \lfloor\phi\rfloor \longrightarrow \lfloor\phi\rfloor$. The rule WR, where ϕ is the introduced formula, is modelled essentially by post-composing $[]_{\lfloor\phi\rfloor} : 0 \longrightarrow \lfloor\phi\rfloor$. Dually for CL and WL.

Before stating soundness and completeness, we need to clarify our notion of theory. A *signature* Σ consist of a set $\mathcal{A}_\Sigma$ of atomic formulæ and a set $\mathcal{K}_\Sigma$ of *non-logical axioms*, which are atomic derivations of the form $\dfrac{}{\Gamma \vdash \Delta}\,c$ with no hypotheses. (Although non-logical axioms can lead to logical inconsistency, e.g., if $\Gamma = \top$ and $\Delta = \bot$, they are necessary just as λ-terms of "false" type are necessary to form the internal language of a cartesian closed category.)

Definition 3.5. A *sequent theory* $\mathcal{T}$ over a signature Σ is a set of inequalities of the form $\Phi \preccurlyeq \Psi$, where both Φ and Ψ are proofs of the same sequent $\Gamma \vdash \Delta$ over Σ, such that:

1. The relation $\preccurlyeq$ is reflexive, transitive, and compatible (i.e., all inference rules are "monotonic" w.r.t. $\preccurlyeq$);
2. The relation $\preccurlyeq$ contains all instances of the well-known rules for eliminating logical cuts, forwards *and* backwards, e.g., the one below (for details, see [4]);

$$\Delta\nabla \qquad \begin{array}{ccc} B\oplus A\oplus C & \xrightarrow{\quad\Delta\quad} & (B\oplus A\oplus C)\otimes(B\oplus A\oplus C) \\ {\scriptstyle id\oplus\Delta\oplus id}\big\downarrow & \leq & \big\downarrow{\scriptstyle \delta_2} \\ B\oplus(A\otimes A)\oplus C & \xleftarrow[\nabla\oplus id\oplus\nabla]{} & B\oplus B\oplus(A\otimes A)\oplus C\oplus C \end{array} \qquad \begin{array}{ccc} B\oplus A\oplus C & \xrightarrow{\quad\langle\rangle\quad} & 1 \\ {\scriptstyle id\oplus\langle\rangle\oplus id}\big\downarrow & \leq & \big\downarrow{\scriptstyle\cong} \\ B\oplus 1\oplus C & \xleftarrow[{[]\oplus id\oplus []}]{} & 0\oplus 1\oplus 0 \end{array} \qquad \langle\rangle[]$$

$$\nabla\Delta \qquad \begin{array}{ccc} B\otimes A\otimes C & \xleftarrow{\quad\nabla\quad} & (B\otimes A\otimes C)\oplus(B\otimes A\otimes C) \\ {\scriptstyle id\otimes\nabla\otimes id}\big\uparrow & \leq & \big\uparrow{\scriptstyle \delta_2'} \\ B\otimes(A\oplus A)\otimes C & \xrightarrow[\Delta\otimes id\otimes\Delta]{} & B\otimes B\otimes(A\oplus A)\otimes C\otimes C \end{array} \qquad \begin{array}{ccc} B\otimes A\otimes C & \xleftarrow{\quad[]\quad} & 0 \\ {\scriptstyle id\otimes []\otimes id}\big\uparrow & \leq & \big\uparrow{\scriptstyle\cong} \\ B\otimes 0\otimes C & \xrightarrow[\langle\rangle\otimes id\otimes\langle\rangle]{} & 1\otimes 0\otimes 1 \end{array} \qquad []\langle\rangle$$

Table 1: Inequalities of a classical category

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma\vdash\phi\end{matrix} \quad \begin{matrix}\Psi\\ \vdots\\ \Gamma'\vdash\psi\end{matrix}}{\Gamma,\Gamma'\vdash\phi\wedge\psi} \qquad \dfrac{\begin{matrix}\Xi\\ \vdots\\ \phi,\psi\vdash\Delta\end{matrix}}{\phi\wedge\psi\vdash\Delta}}{\Gamma,\Gamma'\vdash\Delta} \quad \begin{matrix}\preccurlyeq\\ \succcurlyeq\end{matrix} \quad \dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma\vdash\phi\end{matrix} \qquad \dfrac{\begin{matrix}\Psi\\ \vdots\\ \Gamma'\vdash\psi\end{matrix} \quad \begin{matrix}\Xi\\ \vdots\\ \phi,\psi\vdash\Delta\end{matrix}}{\phi,\Gamma'\vdash\Delta}}{\Gamma,\Gamma'\vdash\Delta}$$

3. The relation $\preccurlyeq$ contains the two rules in Table 2, and their duals RedWR and RedCR (*not* backwards);
4. The standard axiom expansions (a.k.a. η-rules) hold forwards and backwards, as well as certain coherence laws for ER, CR and WR, and for EL, CL, and WL (for details, see [4]).

Theorem 3.6 (Cut-elimination). *For every proof Φ in a sequent theory without non-logical constants, there is a cut-free proof Φ' such that $\Phi \preccurlyeq \Phi'$.*

A striking aspect of sequent theories is that the reductions for logical cuts are required to hold backwards (i.e., redex and reduct are equivalent), but not so the reductions for weakening and contraction. That the former hold backwards is in harmony with usual semantics of multiplicative linear logic in symmetric linearly distributive categories.

That RedWL and RedWR cannot both hold backwards follows immediately from Lafont's example: if they held backwards, then the reducts Φ_1 and Φ_2 in Lafont's example would be equivalent to the redex and therefore to each other. Thus, any two proofs of $\Gamma \vdash \Delta$ would be equivalent, and any model would have to collapse.

The situation for RedCL and RedCR is subtle: in § 5, we shall give an example that shows why they cannot hold backwards in our GoI models.

Theorem 3.7 (Soundness). *Let $\mathcal{T}$ be a set of proofs over a signature Σ. Then for every interpretation $\mathbf{C}\lfloor - \rfloor$ of $\mathcal{T}$ in a classical category $\mathbf{C}$, the judgements $\Phi \preccurlyeq \Psi$ such that $\lfloor\Phi\rfloor \leq \lfloor\Psi\rfloor$ form a sequent theory.*

Proof. The soundness of RedCL follows from the lax naturality of Δ and the law $\Delta\nabla$. The lax naturality models the fact that the subproof Φ is copied, while the law $\Delta\nabla$ models the fact that the CL with principal formula ϕ is "replaced" by the CR's for Δ_1 and Δ_3.

Similarly, the soundness of RedWL follows from the lax naturality and the law $\langle\rangle[]$. The lax naturality models the fact that the subproof Φ is discarded, while the law $\langle\rangle[]$ models the fact that the WL that introduces ϕ is "replaced" by the WR's that introduce Δ_1 and Δ_3.

For a detailed proof, see [4]. ☐

Theorem 3.8 (Completeness). *Let $\mathcal{T}$ be a sequent theory, and suppose that for proofs Φ and Ψ of $\Gamma \vdash \Delta$ we have $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ in every interpretation $\mathbf{C}\lfloor - \rfloor$ in a classical category $\mathbf{C}$. Then the judgement $\Phi \preccurlyeq \Psi$ is in $\mathcal{T}$.*

Proof. Via a term model construction; details in [4]. ☐

We have mentioned above that axiom expansions (η-rules) are taken in sequent theories. Indeed, soundness and completeness are proved with respect to these laws (and laws of this type are normally validated by categorical models of proof systems [9, 3, 2]). It may be argued, however, that axiom expansions should not be included in sequent theories. To see this, consider that we might choose to identify proofs with their sets of normal forms. But, as Edmund Robinson has pointed out, the axiom-expanded version of a proof may have a larger set of normal forms than the original proof. Hyland [7] briefly discusses this issue.

4. Classical categories from quantaloids

Next, we show that the compact closed category $\mathcal{G}(\mathbf{Q})$ forms a classical category for every quantaloid $\mathbf{Q}$ with finite biproducts. The proposition below, whose proof is routine, is a generalization of Proposition 6.3 in [8]:

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma_2 \vdash \Delta_1, \phi, \Delta_3\end{array} \qquad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma_1, \Gamma_3 \vdash \Delta_2\end{array}}{\Gamma_1, \phi, \Gamma_3 \vdash \Delta_2}\,\mathsf{WL}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,\mathsf{Cut} \quad \preccurlyeq \quad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma_1, \Gamma_3 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,\mathsf{WL}, \mathsf{WR} \qquad \textsc{Red}\mathsf{WL}$$

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma_2 \vdash \Delta_1, \phi, \Delta_3\end{array} \qquad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma_1, \phi, \phi, \Gamma_3 \vdash \Delta_2\end{array}}{\Gamma_1, \phi, \Gamma_3 \vdash \Delta_2}\,\mathsf{CL}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,\mathsf{Cut} \quad \preccurlyeq \quad \dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma_2 \vdash \Delta_1, \phi, \Delta_3\end{array} \qquad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma_2 \vdash \Delta_1, \phi, \Delta_3\end{array} \qquad \begin{array}{c}\Psi\\ \vdots\\ \Gamma_1, \phi, \phi, \Gamma_3 \vdash \Delta_2\end{array}}{\Gamma_1, \phi, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,\mathsf{Cut}}{\Gamma_1, \Gamma_2, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_1, \Delta_2, \Delta_3, \Delta_3}\,\mathsf{Cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,\mathsf{CL}, \mathsf{CR} \qquad \textsc{Red}\mathsf{CL}$$

Table 2: Cut-reductions for weakening and contraction (representative cases)

Proposition 4.1. *For every quantaloid* $\mathbf{Q}$ *with finite biproducts, the operation that sends a morphism*

$$f = \begin{pmatrix} f_{AB} & f_{XB} \\ f_{AX} & f_{XX} \end{pmatrix} : A \oplus X \longrightarrow B \oplus X$$

to $f_{AB} \sqcup f_{BX} \circ f_{XX}^{*} \circ f_{AX} : A \longrightarrow B$ *is a trace operator.*

It is helpful to spell out the structure of the category $\mathcal{G}(\mathbf{Q})$ in terms of the structure of $\mathbf{Q}$. By definition, a morphism $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ of $\mathcal{G}(\mathbf{Q})$ is a morphism $f : A^+ \oplus B^- \longrightarrow A^- \oplus B^+$ of $\mathbf{Q}$. Because $\oplus$ is a biproduct, f can be given by four morphisms:

$$\begin{array}{ccc} A^+ & \xrightarrow{\;f_{AB}\;} & B^+ \\ {\scriptstyle f_{AA}}\downarrow & & \uparrow{\scriptstyle f_{BB}} \\ A^- & \xleftarrow[\;f_{BA}\;]{} & B^- \end{array} \qquad (1)$$

For the composition $g \circ f$, it is helpful to arrange the components of f and g in a diagram:

$$\begin{array}{ccccc} A^+ & \xrightarrow{\;f_{AB}\;} & B^+ & \xrightarrow{\;g_{BC}\;} & C^+ \\ {\scriptstyle f_{AA}}\downarrow & & {\scriptstyle f_{BB}}\uparrow\;\downarrow{\scriptstyle g_{BB}} & & \uparrow{\scriptstyle g_{CC}}\,. \\ A^- & \xleftarrow[\;f_{BA}\;]{} & B^- & \xleftarrow[\;g_{CB}\;]{} & C^- \end{array} \qquad (2)$$

The components of $g \circ f$, which can be obtained directly from Diagram 2 by chasing paths, are

$$\begin{aligned}
(g \circ f)_{AC} &= g_{BC} \circ (f_{BB} \circ g_{BB})^* \circ f_{AB} \\
(g \circ f)_{AA} &= f_{AA} \sqcup f_{BA} \circ (g_{BB} \circ f_{BB})^* \circ g_{BB} \circ (f_{BB} \circ g_{BB})^* \circ f_{AB} \\
(g \circ f)_{CC} &= g_{CC} \sqcup g_{BC} \circ (f_{BB} \circ g_{BB})^* \circ f_{BB} \circ (g_{BB} \circ f_{BB})^* \circ g_{CB} \\
(g \circ f)_{CA} &= f_{BA} \circ (g_{BB} \circ f_{BB})^* \circ g_{CB}
\end{aligned}$$

(Diagram 2 was in fact a starting point for the GoI work presented in this article; the need for the $(-)^*$ operator inspired the use of quantaloids, and we realized only later that Diagram 2 amounted to GoI.)

The identity on (A^+, A^-) is given by the square whose two horizontal arrows are identities and whose two vertical arrows are zeros. For a detailed description of the remaining structure of $\mathcal{G}(\mathbf{Q})$, see [4].

Before stating the following theorem, we need some definitions. A morphism $f : A \longrightarrow B$ is called a *semigroup homomorphism* if it preserves multiplication — that is, $f \circ \nabla = \nabla \circ (f \oplus f)$ (note that the $\leq$ direction always holds); f is called a *pointed homomorphism* if it preserves the unit — that is $f \circ [] = []$; f is called a *monoid homomorphism* if both conditions hold. Dually for *co-semigroup homomorphism*, *co-pointed homomorphism*, and *co-monoid homomorphism*. As we shall see (Proposition 4.6), whether the denotation f of a proof is a (co-)monoid homomorphism is closely linked with the proof's use of negation.

Theorem 4.2. *For every quantaloid* $\mathbf{Q}$ *with finite biproducts, the compact closed category* $\mathcal{G}(\mathbf{Q})$ *forms a classical category.*

Moreover, for all $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ *in* $\mathcal{G}(\mathbf{Q})$*, the following are equivalent: (1)* $f_{AA} = 0$*, (2)* f *is a semigroup-homomorphism, (3)* f *is a co-pointed homomorphism.*

Dually, the following are equivalent: (1) $f_{BB} = 0$*, (2)* f *is a co-semigroup-homomorphism, (3)* f *is a pointed homomorphism.*

Consequently, the following are equivalent: (1) $f_{AA} = 0$ *and* $f_{BB} = 0$*, (2)* f *is a monoid-homomorphism, (3)* f *is a co-monoid-homomorphism.*

Proof. For morphisms $f, g : (A^+, A^-) \longrightarrow (B^+, B^-)$ in $\mathcal{G}(\mathbf{Q})$, we define $f \leq g$ if and only if $g_{AA} \sqsubseteq f_{AA}$ and $g_{BA} \sqsubseteq f_{BA}$ and $g_{AB} \sqsubseteq f_{AB}$ and $g_{BB} \sqsubseteq f_{BB}$. The monoid on (A^+, A^-) is given as below, and dually for the co-monoid. For a detailed proof, see [4]. □

$$\nabla_{(A^+,A^-)} = \begin{array}{ccc} A^+ \oplus A^+ & \xrightarrow{\nabla} & A^+ \\ 0\big\downarrow & & 0\big\uparrow \\ A^- \oplus A^- & \xleftarrow{\Delta} & A^- \end{array} \qquad [\,]_{(A^+,A^-)} = \begin{array}{ccc} 0 & \xrightarrow{[\,]} & A^+ \\ 0\big\downarrow & & 0\big\uparrow \\ 0 & \xleftarrow{\langle\rangle} & A^- \end{array}$$

4.1. The matrix presentation of $\mathcal{G}(\mathbf{Q})$

We shall explain how to describe the denotations of proofs in a category $\mathcal{G}(\mathbf{Q})$ as matrices.

Let $\mathbf{C}$ be any classical category with $\otimes \cong \oplus$. Proofs of sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$ denote morphisms $\lfloor\phi_1\rfloor \oplus \cdots \oplus \lfloor\phi_n\rfloor \longrightarrow \lfloor\psi_1\rfloor \oplus \cdots \oplus \lfloor\psi_m\rfloor$. Proofs of sequents of the form

$$\Gamma, \phi \wedge \psi, \Gamma' \vdash \Delta \qquad \text{or} \qquad \Gamma, \phi \vee \psi, \Gamma' \vdash \Delta$$

denote morphisms $\lfloor\Gamma\rfloor \oplus \lfloor\phi\rfloor \oplus \lfloor\psi\rfloor \oplus \lfloor\Gamma'\rfloor \longrightarrow \lfloor\Delta\rfloor$. Dually, proofs of sequents of the form

$$\Gamma \vdash \Delta, \phi \wedge \psi, \Delta' \qquad \text{or} \qquad \Gamma \vdash \Delta, \phi \vee \psi, \Delta'$$

denote morphisms $\lfloor\Gamma\rfloor \longrightarrow \lfloor\Delta\rfloor \oplus \lfloor\phi\rfloor \oplus \lfloor\psi\rfloor \oplus \lfloor\Delta'\rfloor$. Proofs of sequents of the form

$$\Gamma, \top, \Gamma' \vdash \Delta \qquad \text{or} \qquad \Gamma, \bot, \Gamma' \vdash \Delta$$

denote morphisms $\lfloor\Gamma\rfloor \oplus \lfloor\Gamma'\rfloor \longrightarrow \lfloor\Delta\rfloor$, and dually. Proofs of sequents of the form

$$\Gamma, \phi, \Gamma' \vdash \Delta, \Delta' \qquad \text{or} \qquad \Gamma, \Gamma' \vdash, \Delta, \neg\phi, \Delta'$$

denote morphisms $\lfloor\Gamma\rfloor \oplus \lfloor\phi\rfloor \oplus \lfloor\Gamma'\rfloor \longrightarrow \lfloor\Delta\rfloor \oplus \lfloor\Delta\rfloor'$, and dually. So:

Observation 4.3. Proofs of sequents $\Gamma \vdash \Delta$ are interpreted by morphisms $\lfloor p_1^-\rfloor \oplus \cdots \oplus \lfloor p_n^-\rfloor \longrightarrow \lfloor p_1^+\rfloor \oplus \cdots \oplus \lfloor p_m^+\rfloor$, where $\{p_1^-, \ldots, p_n^-, p_1^+, \ldots, p_m^+\}$ is the set of occurrences of atomic formulæ in $\Gamma \vdash \Delta$, and

- each p_i^- is either in Γ under an even number of negations, or in Δ under an odd number of negations (we say that such occurrences of atomic formulæ have *negative polarity*), and
- each p_j^+ is either in Γ under an odd number of negations, or in Δ under an even number of negations (we say that such occurrences have *positive polarity*).

Now for the special case where $\mathbf{C}$ is of the form $\mathcal{G}(\mathbf{Q})$. Let Φ be a proof of $\Gamma \vdash \Delta$, and let p_i^+ and p_j^- be as in Observation 4.3. Because $\lfloor p_i^+\rfloor = (\lfloor p_i^+\rfloor^+, \lfloor p_i^+\rfloor^-)$ and $\lfloor p_i^-\rfloor = (\lfloor p_i^-\rfloor^+, \lfloor p_i^-\rfloor^-)$, Φ denotes essentially a morphism

$$\begin{aligned} &(\lfloor p_1^-\rfloor^+ \oplus \cdots \oplus \lfloor p_n^-\rfloor^+, \lfloor p_1^-\rfloor^- \oplus \cdots \oplus \lfloor p_n^-\rfloor^-) \\ &= \lfloor p_1^-\rfloor \oplus \cdots \oplus \lfloor p_n^-\rfloor \longrightarrow \lfloor p_1^+\rfloor \oplus \cdots \oplus \lfloor p_m^+\rfloor \\ &= (\lfloor p_1^+\rfloor^+ \oplus \cdots \oplus \lfloor p_n^+\rfloor^+, \lfloor p_1^+\rfloor^- \oplus \cdots \oplus \lfloor p_n^+\rfloor^-) \end{aligned}$$

of $\mathcal{G}(\mathbf{Q})$. For simplicity, we assume from here on that the denotations of atomic formulæ p are *positive*, in the sense that $\lfloor p\rfloor^- = 0$. (Intuitively, this corresponds to the fact that atomic formulæ have no subformulæ under an odd number of negations — because they have no subformulæ at all.) So every proof of $\Gamma \vdash \Delta$ denotes a morphism

$$(\lfloor p_1^-\rfloor^+ \oplus \cdots \oplus \lfloor p_n^-\rfloor^+, 0) \longrightarrow (\lfloor p_1^+\rfloor^+ \oplus \cdots \oplus \lfloor p_m^+\rfloor^+, 0)$$

of $\mathcal{G}(\mathbf{Q})$, which is given by a morphism

$$f : \lfloor p_1^-\rfloor^+ \oplus \cdots \oplus \lfloor p_n^-\rfloor^+ \longrightarrow \lfloor p_1^+\rfloor^+ \oplus \cdots \oplus \lfloor p_m^+\rfloor^+$$

of $\mathbf{Q}$. This in turn corresponds to a matrix

$$f = \begin{array}{c|ccc} & p_1^- & \cdots & p_n^- \\ \hline p_1^+ & f_{11} & \cdots & f_{n1} \\ \vdots & \vdots & & \vdots \\ p_m^+ & f_{m1} & \cdots & f_{nm} \end{array}$$

where $f_{ij} \in \mathbf{Q}(\lfloor p_i\rfloor^-, \lfloor p_j\rfloor^+)$. We call f_{ij} the *data-flow* from p_i^- to p_j^+, and f the *data-flow matrix*.

4.2. A graphical presentation of data-flow

Next, we describe the interpretation of proofs in a classical category $\mathbf{C}$ for the special case where $\mathbf{C}$ has the form $\mathcal{G}(\mathbf{Q})$. Guided by our insights from § 4.1, we assume that the interpretations of atomic formulæ are positive. Thus, to describe the interpretation is to describe the data-flow matrix of any given proof.

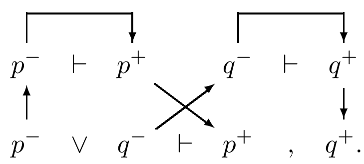
As it turns out, the matrices are best described in a graphical way. This graphical description is presented in Table 3; next, we shall explain how it describes matrices. As a first example, consider the proof below, where p and q are atomic, and occurrences of formulæ are annotated with $+$ and $-$ according to their polarities.

$$\dfrac{\dfrac{}{p^- \vdash p^+}\,\mathsf{Ax} \qquad \dfrac{}{q^- \vdash q^+}\,\mathsf{Ax}}{p^- \vee q^- \vdash p^+, q^+}\,\vee\mathsf{L}$$

By Observation 4.3, we need a matrix of the form

$$\begin{array}{c|cc} & p^- & q^- \\ \hline p^+ & f_{11} & f_{21} \\ q^+ & f_{12} & f_{22} \end{array}$$

Decorating the proof according to Table 3 yields



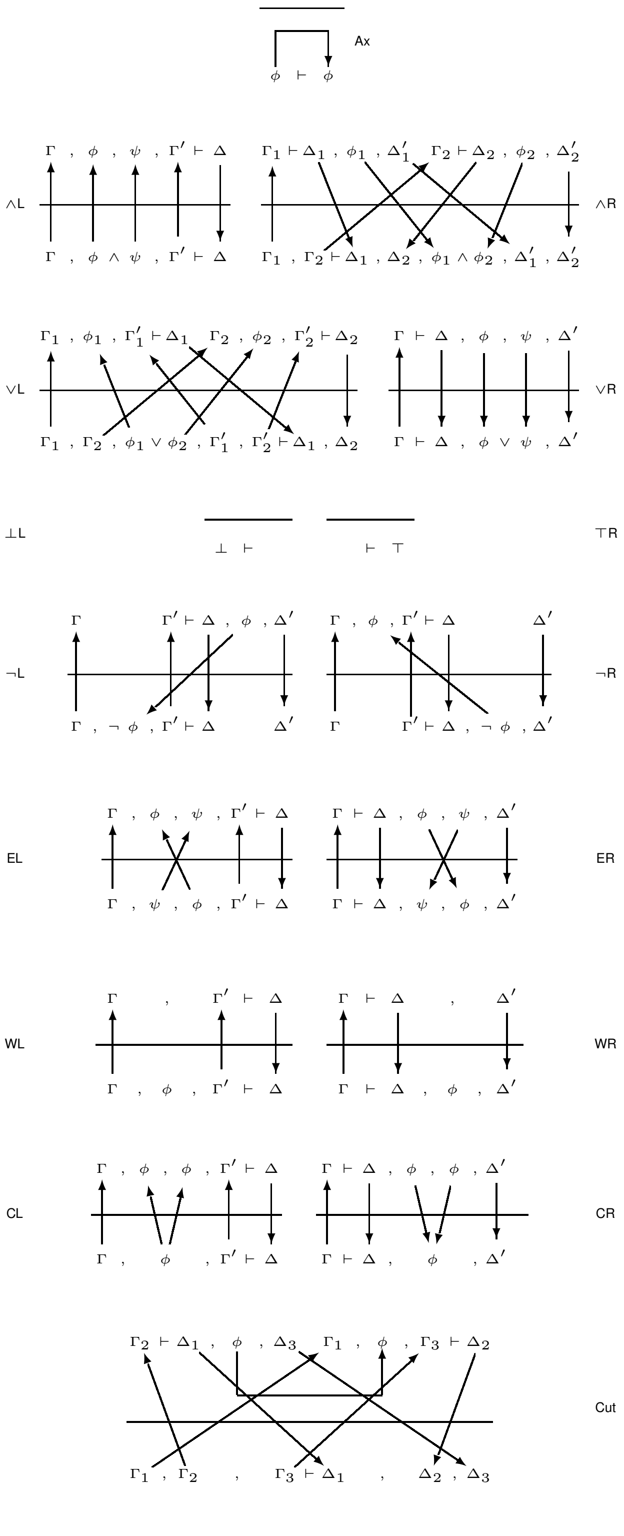


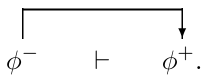
Table 3: Data-flow in sequent proofs

There is a path from the bottom p^- to the bottom p^+ and from the bottom q^- to the bottom q^+. Those two paths signify that $f_{11} = id$ and $f_{22} = id$. There is neither a path from the bottom p^- to the bottom q^+, nor from the bottom q^- and the bottom p^+. This signifies that $f_{21} = 0$ and $f_{12} = 0$.

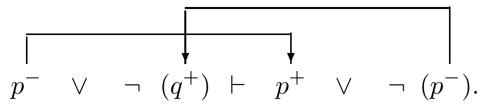
To understand negation, consider the proof

$$\frac{}{\phi^- \vdash \phi^+}\ \text{Ax},$$

where $\phi = p \vee \neg q$ and p, q are atomic and the occurrences of ϕ are annotated according to their polarity. Decorating the proof according to Table 3 yields

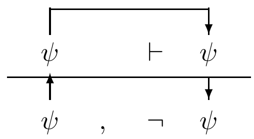


This is supposed to stand for

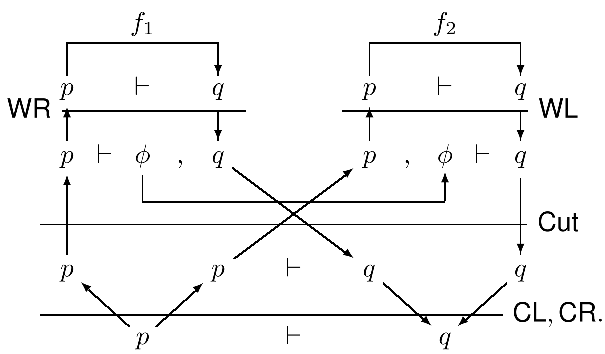


That is, an arrow from ϕ^- to ϕ^+ stands for a bundle of arrows, comprising one arrow from each negative occurrence of an atomic subformula p of ϕ^+ resp. ϕ^- to the corresponding positive occurrence of p in ϕ^- resp. ϕ^+. Thus we have a 2×2-matrix with $f_{11} = id$, $f_{22} = id$, $f_{21} = 0$, and $f_{12} = 0$.

Negation can cause data-flow within one side of a sequent. For example, consider the proof below, which has a data-flow whose source and target are on the left side of $\vdash$.



For another example, consider the Lafont proof with $\Gamma = p$ and $\Delta = q$, where p and q are atomic. Suppose that the 1×1-matrices denoted by Φ_1 and Φ_2 are (f_1) and (f_2), respectively (so f_1 and f_2 are morphisms $\lfloor p \rfloor \longrightarrow \lfloor q \rfloor$). Decorating the proof yields the graph below.



There are two paths from the bottom-left p to the bottom-right q, namely $id_{\lfloor q \rfloor} \circ id_{\lfloor q \rfloor} \circ id_{\lfloor q \rfloor} \circ f_i \circ id_{\lfloor p \rfloor} \circ id_{\lfloor p \rfloor} \circ$

$id_{\lfloor p \rfloor} = f_i$ for $i \in \{1, 2\}$. The data-flow from p to q in the whole proof is defined to be the join of all paths from p to q. So its 1×1 data-flow matrix is $f_1 \sqcup f_2$.

Now we are ready to explain Table 3 in a formal way:

- An arrow $\Gamma \longrightarrow \Gamma$, where $\Gamma = \{\phi_1, \ldots \phi_n\}$, stands for a bundle of arrows $\phi_1 \longrightarrow \phi_1, \ldots, \phi_n \longrightarrow \phi_n$;
- An arrow $\phi \vee \psi \longrightarrow \phi \vee \psi$ or $\phi \wedge \psi \longrightarrow \phi \wedge \psi$ stands for two arrows $\phi \longrightarrow \phi$ and $\psi \longrightarrow \psi$;
- An arrow $\neg\phi \longrightarrow \neg\phi$ stands for an arrow $\phi \longleftarrow \phi$ (the direction matters, because arrows go between *occurrences* of formulæ);
- Thus, it follows inductively that every arrow stands for a bundle of arrows between atomic formulæ;
- Every arrow from atomic formula p to atomic formula q is labelled with a morphism $\lfloor p \rfloor \longrightarrow \lfloor q \rfloor$. If the label is missing, we mean the identity morphism;
- If there are m paths from p to q, and the i-th path has the form $p = p_0^i \xrightarrow{f_1^i} p_1^i \xrightarrow{f_2^i} \cdots \xrightarrow{f_n^i} p_{n_i}^i = \lfloor q \rfloor$, then the entry f_{pq} of the data-flow matrix is

$$\bigsqcup_{i \in \{1, \ldots m\}} f_{n_i}^i \circ f_{n_{i-1}}^i \circ \cdots \circ f_1^i.$$

Remark 4.4. In the absence of non-logical constants, all data-flows between atomic formulæ are id or 0; however, denotations of non-logical constants can be chosen freely, thus more exotic matrices can arise.

Next, we show how the data-flow view obviates a striking connection between negation and the question whether a denotation of a proof in $\mathcal{G}(\mathbf{Q})$ is a (co-)monoid homomorphism (cf. Theorem 4.2).

Proposition 4.5. *Let $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ be the denotation of some proof Φ of $\Gamma \vdash \Delta$ in $\mathcal{G}(\mathbf{Q})$. Let f_{AA}, f_{AB}, f_{BA}, and f_{BB} be as in Diagram 1. Then*

- *f_{AA} is given by the data-flow in Φ from Γ to Γ.*
- *f_{AB} is given by the data-flow in Φ from Γ to Δ.*
- *f_{BA} is given by the data-flow in Φ from Δ to Γ.*
- *f_{BB} is given by the data-flow in Φ from Δ to Δ.*

Proof. By induction on Φ. □

Proposition 4.6. *Let $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ be the denotation of some proof Φ of $\Gamma \vdash \Delta$ in $\mathcal{G}(\mathbf{Q})$. If Φ contains neither $\neg$L, nor $\neg$R, nor non-logical axioms, then $f_{AA} = 0$ and $f_{BB} = 0$; in other words, f is a (co-)monoid homomorphism*

Proof. By induction on Φ, it follows Φ has no data-flow from Γ to Γ or from Δ to Δ. Now the claim follows from Proposition 4.5. □

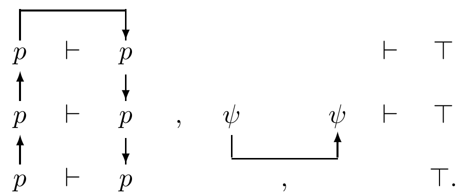
5. Weakening and contraction in $\mathcal{G}(\mathbf{Q})$

Next, we study the cut-reduction rules for weakening and contraction, which are the only semantic troublemakers. We focus on explaining why denotations of proofs in $\mathcal{G}(\mathbf{Q})$ can grow *strictly* along the rules RedWL and RedCL (the explanation for RedWR and RedCR is dual).

For weakening, consider the following instance of RedWL, where p is atomic and ψ any formula:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\mathsf{Ax}}{p \vdash p, \psi}\,\mathsf{WR} \qquad \dfrac{\dfrac{}{\vdash \top}\,\top\mathsf{R}}{\psi \vdash \top}\,\mathsf{WL}}{p \vdash p, \top}\,\mathsf{Cut} \;\preccurlyeq\; \dfrac{\dfrac{}{\vdash \top}\,\top\mathsf{R}}{p \vdash p, \top}\,\mathsf{WL,WR}.$$

The data-flow graph for the redex is



By contrast, the data-flow graph for the reduct contains no arrows at all. So, in the redex, the data-flow from the negative p to the positive p is id, whereas in the reduct it is 0. Thus, the redex is *greater* w.r.t. the local order $\leq$ of the classical category $\mathcal{G}(\mathbf{Q})$.

For contraction, consider the instance of RedCL where

$$\Phi = \dfrac{\dfrac{\dfrac{}{p \vdash p}}{\vdash \neg p, p}}{\vdash \neg p \vee p} \qquad \Psi = \dfrac{\dfrac{}{\neg p \vee p \vdash \neg p \vee p} \qquad \dfrac{}{\neg p \vee p \vdash \neg p \vee p}}{\neg p \vee p, \neg p \vee p \vdash (\neg p \vee p) \wedge (\neg p \vee p)}\,.$$

The graphs for the redex and reduct are presented in Figures 2 and 3, respectively. To help discussion, let us decorate the conclusion of redex and reduct:

$$\vdash (\neg(p_1^-) \vee p_1^+) \wedge (\neg(p_2^-) \vee p_2^+).$$

The data-flow matrices of the redex and the reduct

	p_1^-	p_2^-
p_1^+	id	id
p_2^+	id	id

and

	p_1^-	p_2^-
p_1^+	id	0
p_2^+	0	id

,

respectively. So the reduct is greater than the redex in the sense of the order $\leq$ of the classical category.

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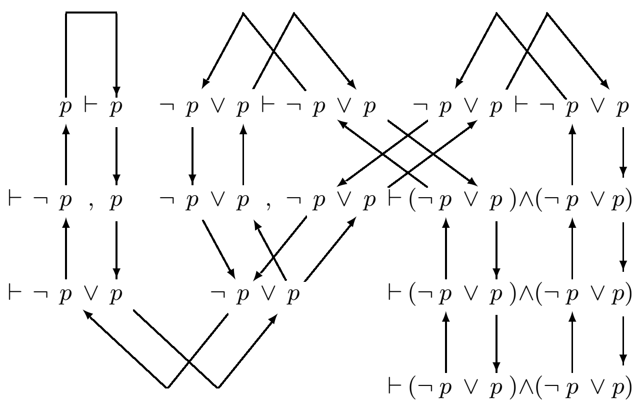

Figure 2: Instance of the redex of REDCL

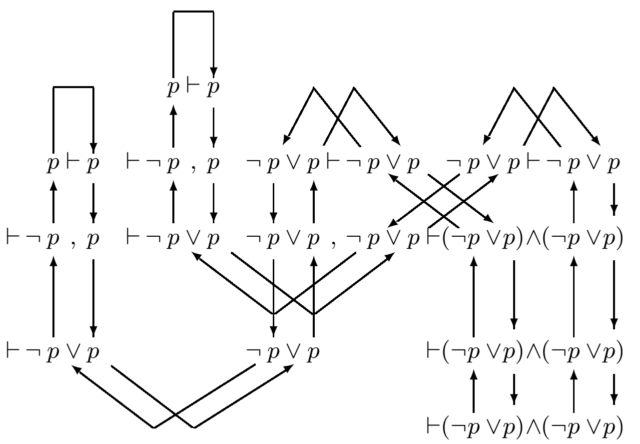

Figure 3: Instance of the reduct of REDCL

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