On Categorical Models of Classical Logic and the Geometry of Interaction

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December 22, 2004

Abstract

It is well-known that weakening and contraction cause naïve categorical models of the classical sequent calculus to collapse to Boolean lattices. In previous work, summarized briefly herein, we have provided a class of models called *classical categories* which is sound and complete and avoids this collapse by interpreting cut-reduction by a poset-enrichment. Examples of classical categories include boolean lattices and the category of sets and relations, where both conjunction and disjunction are modelled by the set-theoretic product.

In this article, which is self-contained, we present an improved axiomatization of classical categories, together with a deep exploration of their structural theory. Observing that the collapse already happens in the absence of negation, we start with negation-free models called *Dummett categories*. Examples include, besides the classical categories above, the category of sets and relations, where both conjunction and disjunction are modelled by the disjoint union. We prove that Dummett categories are MIX, and that the partial order can be derived from hom-semilattices which have a straightforward proof-theoretic definition. Moreover, we show that the Geometry-of-Interaction construction can be extended from multiplicative linear logic to classical logic, by applying it to obtain a classical category from a Dummett category.

Along the way, we gain detailed insights into the changes that proofs undergo during cut elimination in the presence of weakening and contraction.

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1 Introduction

It is notoriously hard to find a decent denotational semantics for the classical sequent calculus, let alone an algorithmic interpretation. This problem is related to the *non-deterministic* behaviour of cut elimination. To see the point, consider the following sequent proof:

$$\Lambda \quad = \quad \cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}\Phi_1\\ \vdots\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash A, \Delta}\text{weakening} \qquad \cfrac{\begin{matrix}\Phi_2\\ \vdots\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma, A \vdash \Delta}\text{weakening}}{\Gamma, \Gamma \vdash \Delta, \Delta}\mathsf{Cut}}{\text{contractions}}}{\Gamma \vdash \Delta} ,$$

where Φ_1 and Φ_2 are arbitrary proofs of the sequent $\Gamma \vdash \Delta$. We call this the "Lafont proof", because it is a variant of an example credited to Lafont (cf. (Girard, Lafont & Taylor 1989, p. 151)). The sub-proof Φ_1 is weakened on the right, and the sub-proof Φ_2 is weakened on the left. Then follows a cut, where the cut formula is the formula A introduced by the weakenings. Finally, the double occurrences of Γ and Δ are removed by left and right contractions. (Clearly, the two contractions are supposed to commute with each other, so we need not be specific about the order in which they are applied.) The proof Λ

reduces to

$$\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma \vdash \Delta \\ \hline \text{weakenings} \\ \hline \Gamma,\Gamma\vdash\Delta,\Delta \\ \hline \text{contractions}\\ \hline \Gamma\vdash\Delta \end{array} \qquad \text{or to} \qquad \begin{array}{c}\Phi_2\\ \vdots\\ \Gamma \vdash \Delta \\ \hline \text{weakenings} \\ \hline \Gamma,\Gamma\vdash\Delta,\Delta \\ \hline \text{contractions}\\ \hline \Gamma\vdash\Delta \end{array}.$$

But, clearly, the weakenings followed by the contractions are essentially nothing (cf. (Girard et al. 1989, p. 152)). So Φ_1 and Φ_2 are obtained by reducing the same proof. Thus, the denotations of Φ_1 and Φ_2 must be equal for any semantics that *admits cut-reduction* in the sense that the reduct is denotationally equal to the redex. In summary, any denotational semantics that admits cut-reduction must identify all proofs of a sequent $\Gamma \vdash \Delta$. Note that this argument does not rely on negation!

There are various escapes from this denotational collapse: first, we might simply abandon classical logic and adopt, for example, intuitionistic logic or linear logic instead. As explained in Gentzen's seminal article (Gentzen 1934), intuitionistic logic can an be obtained by restricting the classical sequent calculus is such a way that the succedent Δ contains at most one formula. As is widely known, intuitionistic logic can be modelled by cartesian-closed categories. Models of linear logic also abound. But both intuitionistic logic and linear logic differ from classical logic with respect to provable sequents, and we do not wish to depart from classical provability.

Second, insisting to keep classical logic, we might move to "classical natural deduction" systems (Prawitz 1965), where proofs may be represented as terms of the $\lambda\mu\nu$-calculus (Parigot 1992, Pym & Ritter 2001). But such systems do not admit all cut-reductions: as it turns out, the call-by-name version of $\lambda\mu\nu$ admits only the reduction to Φ_2, while the call-by-value version admits only the reduction to Φ_1. Each version corresponds to a different choice of $\neg\neg$-translations (a.k.a. "continuation-passing-style transforms" in programming-language jargon) of classical logic into intuitionistic logic (Troelstra & Schwichtenberg 1996, Plotkin 1975). Models of $\lambda\mu\nu$ can be obtained in fibrations over a base category of structural maps in which each fibre is a model of intuitionistic natural deduction and in which dualizing negation is interpreted as certain maps between the fibres (Ong 1996, Pym & Ritter 2001). Alternative models are given by control categories and co-control categories (Selinger 2001).

In our companion paper (Führmann & Pym 2004*b*), we presented a solution that, unlike classical natural deduction, models *all* cut-reductions: we introduced a kind of poset-enriched category called *classical category* whose objects model types and whose morphisms model proofs of the classical sequent calculus; whenever a proof of Φ can be reduced to another proof Ψ, we only require $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ (as opposed to $\mathbf{C}\lfloor\Phi\rfloor = \mathbf{C}\lfloor\Psi\rfloor$), where $\mathbf{C}\lfloor\Phi\rfloor$ and $\mathbf{C}\lfloor\Psi\rfloor$ are the morphisms denoted by Φ and Ψ in the classical category $\mathbf{C}$. Classical categories are a special case of symmetric linearly distributive categories (Cockett & Seely 1997*b*): they have symmetric monoidal products $\otimes$ and $\oplus$ for modelling conjunction and disjunction, respectively. To model contraction and weakening on the right, every object A is endowed with a symmetric monoid $(\blacktriangledown_A : A \oplus A \longrightarrow A, []_A : \bot \longrightarrow A)$; the multiplication $\blacktriangledown$ models contraction, and the unit $[]$ models weakening. Dually for contraction and weakening on the left. (It is worth mentioning here that symmetric linearly distributive categories *with negation* are equivalent to $*$-autonomous categories; however, the former provide better choice of primitives for achieving our goals.)

In (Führmann & Pym 2004*b*), we proved that classical categories are sound and complete for the classical sequent calculus. More precisely, we introduced a notion of theory with judgments of the form

$$\begin{array}{c} \Phi \\ \vdots \\ \Gamma \vdash \Delta \end{array} \quad \preccurlyeq \quad \begin{array}{c} \Psi \\ \vdots \\ \Gamma \vdash \Delta \end{array},$$

where the $\preccurlyeq$ is a preorder that contains all reductions required for cut elimination. The soundness theorem in (Führmann & Pym 2004*b*) states essentially that $\Phi \preccurlyeq \Psi$ implies $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ for every classical category $\mathbf{C}$. The completeness theorem states essentially that $\Phi \preccurlyeq \Phi$ is a theorem of a theory $\mathcal{T}$ whenever $\mathbf{C}\lfloor\Phi\rfloor \leq \mathbf{C}\lfloor\Psi\rfloor$ holds for every model $\mathbf{C}\lfloor-\rfloor$ of $\mathcal{T}$. Its proof uses a category built from Robinson's proof nets for classical logic (Robinson 2003), which correspond directly to the classical sequent calculus. (We shall discuss these nets briefly in § 2.2.) A morphism of that category is an equivalence class of proof nets with respect to the preorder $\preccurlyeq$. For morphisms $f, g : A \longrightarrow B$ with representing nets N_f and N_g, that category has $f \leq g$ if and only if $N_f \preccurlyeq N_g$. (This explains why $\leq$ is a partial order although the preorder $\preccurlyeq$ is not generally antisymmetric.)

In (Führmann & Pym 2004*b*), we gave the following concrete examples of classical categories:

- An initial model built from proof nets;
- The category **Rel** of sets and relation, where both $\otimes$ and $\oplus$ are defined to be the evident functor that takes two sets to their cartesian product, and $\preccurlyeq$ is the set-theoretic inclusion of relations;
- Boolean lattices
- The product of any two classical categories—for example, $\mathbf{Rel} \times \mathbf{B}$ for any Boolean lattice $\mathbf{B}$. This shows that there are models which are non-posetal (i.e., there are hom-sets with more than one element) and non-compact (i.e., $\otimes \neq \oplus$).

In (Führmann & Pym 2004*a*), we found further classical categories that arise from an abstract Geometry-of-Interaction (GoI) construction starting with a quantaloid, and used those models to study the "increase" of denotations during cut elimination.

Since we presented (Führmann & Pym 2004*a*) in the summer of 2004, we managed to (1) greatly advanced the axiomatization and understanding of classical categories, in particular by proving that they are MIX, and (2) strongly generalize the GoI construction we presented in (Führmann & Pym 2004*a*). Many of the new insights were sparked by Masahito Hasegawa in private communications, which is why several propositions in this article are attributed to him.

This article gives a comprehensive account of our improved axiomatization and structural theory of classical categories (§ 3), and of our generalized GoI construction (§ 4). Owing to the substantial advances of presentation and axiomatization, we chose to make this article self-contained and require no previous knowledge of (Führmann & Pym 2004*b*) or (Führmann & Pym 2004*a*).

On the purely technical side, we have adopted the proof nets in the style of (Blute, Cockett, Seely & Trimble 1996); understanding these nets takes a little more effort than understanding Robinson's nets, but they are more efficient for calculations.

1.1 Outline

Now for a detailed overview of this article.

- In § 2, we recall some preliminaries: the classical sequent calculus, proof nets, and the categorical semantics of multiplicative linear logic (MLL) in symmetric linearly distributive categories.

- In § 3, we introduce classical categories from the ground up. We proceed in two steps: first, we extend symmetric linearly distributive categories presented with structure for modelling weakening and contraction. This structure consists of a symmetric monoid and a symmetric comonoid for every object, and a poset-enrichment The resulting categories are models of the negation-free fragment of the classical sequent calculus. We call them *Dummett categories* (inspired by Dummett's extensive discussion, in "Elements of Intuitionism" (Dummett 1977), of multi-succedent intuitionistic sequent calculi). Second, we introduce classical categories as Dummett categories with the property of *having negation* in the sense of Cockett and Seely.

  We then establish the close connection between classical categories and the classical sequent calculus by constructing the free classical category from proof nets (Theorem 3.32). (This extends the construction of the free symmetric linearly distributive category from MLL proof nets in the sense of (Blute et al. 1996).) From a logical point of view, the result means that classical categories are sound and complete (in order-theoretic sense explained above) with respect to a certain super-relation of cut-reduction for the classical sequent calculus.

  Our free construction relies on a series of results about the structure of Dummett categories, including

  - the remarkable result (due to Hasegawa) that the monoids or comonoids cause symmetric linearly distributive categories to be MIX (Theorem 3.11);
  - the fact that the poset-enrichment is not needed as extra structure, but is induced by hom-semilattices which are derivable from other primitives (Prop. 3.28).

  We finish § 3 by presenting a extremely economic axiomatization of compact Dummett categories (Prop. 3.39, due to Hasegawa), and even more economical axiomatization of Dummett categories with finite biproducts (Prop. 3.40, also due to Hasegawa).

- In § 4, we introduce an extended GoI construction that sends a traced Dummett category to a classical category (Theorem 4.4). This shows that GoI works in the presence of weakening and contraction, even with respect to the partial order that models cut-reduction. As we shall explain, traced Dummett categories are essentially traced symmetric monoidal categories, plus symmetric monoids and symmetric comonoids on every object, satisfying certain conditions. Our extended GoI construction is an instance of the well-known construction of a compact closed category from a traced symmetric monoidal category. (For an overview of the history of GoI leading to that construction, see the introduction of § 4.) The key point of our extended construction is that the symmetric monoids and symmetric comonoids, and the conditions required for a Dummett category, "survive" the extended GoI construction.

  In § 4.4, we study the special case where the starting point of the extended GoI construction is a traced Dummett category with finite biproducts. In particular, we present a comprehensive characterization of morphisms in such GoI categories with respect to their behaviour under cut-reduction (Prop. 4.5).

- In § 5, we suggest some directions for future work.

1.2 Related work

The article (Hyland 2004) introduces a notion of *abstract interpretation of classical proof* as a compact closed category in which every object is equipped with a symmetric monoid and a symmetric comonoid, satisfying certain conditions. (This work was foreshadowed in (Hyland 2002).) These abstract interpretations are almost the same as our classical categories in the *compact* case where $\otimes = \oplus$. The only difference is that compact classical categories need to satisfy an extra equation (Equation 3 in § 3.4.2). As we shall show in § 3.4.2, this equation implies that every compact classical category has hom-semilattices, which yield the partial order we use for modelling cut-reduction. So our approach is more general than Hyland's in that it does not require compactness, and more special in that we require certain conditions that lead to the existence of hom-semilattices.

Another overlap with (Hyland 2004) happens where we specialize our GoI construction to categories with finite biproducts. The partial order specific to our models allows a precise analysis of the behaviour of morphisms with respect to cut-reduction (explained in § 3.2.1).

The article (Bellin, Hyland, Robinson & Urban 2004) contains a semantics of the classical sequent calculus which is finer grained than ours in that it rejects axiom expansions (also called η-rules), that is, the categorical connectives $\otimes$ and $\oplus$ which model conjunction and disjunction do not generally preserve identities. In contrast, our work fits into the existing framework of symmetric linearly distributive categories, in which $\otimes$ and $\oplus$ are functorial. Another difference between our work and (Bellin et al. 2004) is that we deal with modelling cut-reduction (using the poset-enrichment) whereas (Bellin et al. 2004) do not. In (Dosen 1999, Dosen & Petric 2004), a notion of "Boolean category" is introduced. This notion relies on the presence of products and coproducts, leading to a more "collapsed" structure than ours, closely related to the category of finite sets and relations.

There is also some interesting work about *confluent* cut elimination in the presence of the MIX rule (Bellin 2003, Lamarche & Straßburger 2004). For example, one can remove the non-determinism of cut-reduction by allowing a reduction

$$\dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma\vdash\Delta\end{array}}{\Gamma\vdash A,\Delta}\,\mathsf{WR} \qquad \dfrac{\begin{array}{c}\Phi'\\ \vdots\\ \Gamma\vdash'\Delta'\end{array}}{\Gamma',A\vdash\Delta'}\,\mathsf{WL}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{Cut} \quad \preccurlyeq \quad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma\vdash\Delta\end{array} \qquad \begin{array}{c}\Phi'\\ \vdots\\ \Gamma\vdash'\Delta'\end{array}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{MIX}.$$

The confluent cut elimination procedure in (Lamarche & Straßburger 2004) (which is based on proof nets) does this implicitly. Our semantics is compatible with this approach: the MIX rule is denotationally equivalent to a degenerate cut with cut formula $A=\bot$ or $A=\top$. (Both choices of A result in the same denotation.) So, in our view, this kind of confluent "cut elimination" is a removal of arbitrary cuts in favour of degenerate cuts (i.e., MIXes); a MIX is still non-deterministic—in fact, it is the pure incarnation of proof-theoretic non-determinism, because it is the "parallel composition" of Φ and Φ' that one might want to reduce to either Φ or Φ'. Our models support this view, because they admit the reduction of MIX to Φ *and* to Φ'. In fact, the hom-semilattices of our models are given by

$$\Phi_1 * \Phi_2 = \dfrac{\dfrac{\dfrac{\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma\vdash\Delta\end{array} \qquad \begin{array}{c}\Phi_2\\ \vdots\\ \Gamma\vdash\Delta\end{array}}{\Gamma,\Gamma\vdash\Delta,\Delta}\,\mathsf{MIX}.}{\text{contractions}}}{\Gamma\vdash\Delta}$$

From a technical point of view, this article is based on *symmetric linearly distributive categories*, which were introduced in (Cockett & Seely 1997*b*). In particular, we heavily use the proof nets for symmetric linearly distributive categories introduced in (Blute et al. 1996), because they are very efficient for the calculations required in this article. We also build on the discussions of MIX categories in (Blute, Cockett & Seely 2000) and (Cockett & Seely 1997*a*), and the notion of traced object in a MIX category presented in (Blute et al. 2000).

We also rely on results from the GoI literature; the related work in this area is described in § 4.

2 Preliminaries

2.1 The sequent calculus

The version of the sequent calculus to which we refer is given in Tables 1 and 2.

We use the system of multiplicative linear logic (MLL) presented in Table 1, plus the rules for weakening and contraction presented in Table 2, and so obtain

$$\frac{}{A \vdash A}\ \mathsf{Ax}$$

$$\wedge\mathsf{L}\quad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, A \wedge B, \Gamma' \vdash \Delta} \qquad \frac{\Gamma_1 \vdash \Delta_1, A_1, \Delta_1' \quad \Gamma_2 \vdash \Delta_2, A_2, \Delta_2'}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_1 \wedge A_2, \Delta_1', \Delta_2'}\quad \wedge\mathsf{R}$$

$$\vee\mathsf{L}\quad \frac{\Gamma_1, A_1, \Gamma_1' \vdash \Delta_1 \quad \Gamma_2, A_2, \Gamma_2' \vdash \Delta_2}{\Gamma_1, \Gamma_2, A_1 \vee A_2, \Gamma_1', \Gamma_2' \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \vee B, \Delta'}\quad \vee\mathsf{R}$$

$$\top\mathsf{L}\quad \frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \top, \Gamma' \vdash \Delta} \qquad \frac{}{\vdash \top}\quad \top\mathsf{R}$$

$$\bot\mathsf{L}\quad \frac{}{\bot \vdash} \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \bot, \Delta'}\quad \bot\mathsf{R}$$

$$\neg\mathsf{L}\quad \frac{\Gamma, \Gamma' \vdash \Delta, A, \Delta'}{\Gamma, \neg A, \Gamma' \vdash \Delta, \Delta'} \qquad \frac{\Gamma, A, \Gamma' \vdash \Delta, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \neg A, \Delta'}\quad \neg\mathsf{R}$$

$$\mathsf{EL}\quad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\quad \mathsf{ER}$$

$$\frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_2 \vdash \Delta_1, \Delta_2, \Delta_3}\ \mathsf{Cut}$$

Table 1: Inference rules of MLL

$$\mathsf{WL}\quad \frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\quad \mathsf{WR}$$

$$\mathsf{CL}\quad \frac{\Gamma, A, A, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\quad \mathsf{CR}$$

Table 2: Inference rules for weakening and contraction

a calculus which differs from LK (Gentzen 1934) only in its use of the multiplicative form of the introduction rules and the absence of implication. We consider implication to be derived—that is, $A \Rightarrow B = \neg A \vee B$. A *sequent* has the form $\Gamma \vdash \Delta$ where Γ and Δ are finite lists of formulæ. The capital Latin letters range over formulæ.

Henceforth, we shall call sequent proofs *derivations*, to avoid clashes with the notion of "proof" at the meta-level.

To facilitate semantics, we shall introduce a more economic version of the sequent calculus just described: the new version is obtained by replacing the rules $\wedge$R, $\vee$L, $\neg$L, $\neg$R, WL, WR, CL, and CR by axioms. For example, to replace $\wedge$R, we introduce an axiom

$$\frac{}{A, B \vdash A \wedge B}\ \mathsf{Ax} \wedge \mathsf{R}$$

and consider $\wedge$R as an abbreviation for

$$\frac{\Gamma_1 \vdash \Delta_1, A_1, \Delta_1' \qquad \dfrac{\Gamma_2 \vdash \Delta_2, A_2, \Delta_2' \qquad \dfrac{}{A_1, A_2 \vdash A_1 \wedge A_2}\ \mathsf{Ax} \wedge \mathsf{R}.}{A_1, \Gamma_2 \vdash \Delta_2, A_1 \wedge A_2, \Delta_2'}\ \mathsf{Cut}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_1 \wedge A_2, \Delta_1', \Delta_2'}\ \mathsf{Cut}.$$

The extra axioms lead to the revised version of the sequent calculus described in Tables 3, 4, and 5. (We put the rules for negation in a separate table because we shall also study the negation-free fragment.) This revised version facilitates semantics, because axioms simply denote morphisms, and only seven inference rules remain which are not axioms. However, we shall keep the names $\wedge$R, $\vee$L, $\neg$L, $\neg$R, WL, WR, CL, and CR as abbreviations for the evident derivations that involve Ax $\wedge$ R, Ax $\vee$ L, Ax$\neg$L, Ax$\neg$R, AxWL, AxWR, and AxCL, respectively.

For the purpose of categorical logic, we shall consider derivations over any *signature*. A signature Σ consists of a set of atomic formulæ and a set of *optional axioms*. The set of *formulæ over* Σ is generated in the evident way from the atomic formulæ, using $\wedge$, $\vee$, $\top$, $\bot$, and $\neg$. We call a formula over Σ *positive* if it is negation-free. Optional axioms are of the form

$$\frac{}{\Gamma \vdash \Delta}\ f.$$

Typical optional axioms are the ones for weakening and contraction in Table 5.

Definition 2.1. A *derivation* Φ over a signature Σ is a tree generated by the rules in Tables 3 and 4, plus the optional axioms of Σ. We call a derivation over Σ *positive* if all of its formulæ are positive.

2.2 Proof nets

The essence of a sequent proof can be captured by a *proof net*, an idea introduced by Girard (Girard 1987) (or "net" for short). In this article, we shall need proof nets to describe equalities between proofs. The nets we use are, essentially, those from (Blute et al. 1996), extended to account for the additional structural properties of classical logic. Thus in this paper we depart from our use, in (Führmann & Pym 2004*b*), of the classical proof nets introduced by

$$\frac{}{A \vdash A}\ \mathsf{Ax}$$

$$\mathsf{\wedge L}\quad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, A \wedge B, \Gamma' \vdash \Delta} \qquad \frac{}{A, B \vdash A \wedge B}\quad \mathsf{Ax \wedge R}$$

$$\mathsf{Ax \vee L}\quad \frac{}{A \vee B \vdash A, B} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \vee B, \Delta'}\quad \mathsf{\vee R}$$

$$\mathsf{\top L}\quad \frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \top, \Gamma' \vdash \Delta} \qquad \frac{}{\vdash \top}\quad \mathsf{\top R}$$

$$\mathsf{\bot L}\quad \frac{}{\bot \vdash} \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \bot, \Delta'}\quad \mathsf{\bot R}$$

$$\mathsf{EL}\quad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\quad \mathsf{ER}$$

$$\frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_2 \vdash \Delta_1, \Delta_2, \Delta_3}\ \mathsf{Cut}$$

Table 3: Revised inference rules of MLL: negation-free fragment

$$\mathsf{Ax\neg L}\quad \frac{}{\neg A, A \vdash} \qquad \frac{}{\vdash A, \neg A}\quad \mathsf{Ax\neg R}$$

Table 4: Revised inference rules of MLL: axioms for negation

$$\mathsf{AxWL}\quad \frac{}{\bot \vdash A} \qquad \frac{}{A \vdash \top}\quad \mathsf{AxWR}$$

$$\mathsf{AxCL}\quad \frac{}{A \vdash A \wedge A} \qquad \frac{}{A \vee A \vdash A}\quad \mathsf{AxCR}$$

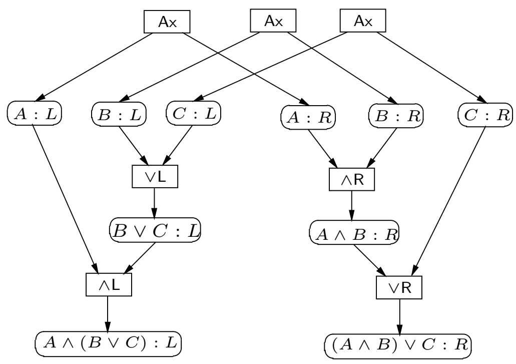
Table 5: Optional axioms for weakening and contraction

Robinson (Robinson 2003). Robinson's nets correspond more directly to the sequent calculus than those of (Blute et al. 1996), the latter being more convenient for calculations.

Informally, a net is a graphical skeleton of a derivation. For example, both derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathsf{Ax} \qquad \dfrac{}{B \vee C \vdash B, C}\,\mathsf{Ax}\vee\mathsf{L}}{A, (B \vee C) \vdash (A \wedge B), C}\,\wedge\mathsf{R}}{A \wedge (B \vee C) \vdash (A \wedge B), C}\,\wedge\mathsf{L}}{A \wedge (B \vee C) \vdash (A \wedge B) \vee C}\,\vee\mathsf{R} \quad \text{and} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash A \wedge B}\,\mathsf{Ax}\wedge\mathsf{R} \qquad \dfrac{}{C \vdash C}\,\mathsf{Ax}}{A, (B \vee C) \vdash (A \wedge B), C}\,\vee\mathsf{L}}{A, (B \vee C) \vdash (A \wedge B) \vee C}\,\vee\mathsf{R}}{A \wedge (B \vee C) \vdash (A \wedge B) \vee C}\,\wedge\mathsf{L}$$

have the following proof net:



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This net is in the style used in (Robinson 2003); in that paper, a *proof structure* is defined to be a bipartite directional graph whose two families of nodes are labelled as follows:

Family 1 labelled by an inference rule of the sequent calculus;

Family 2 labelled by a formula, together with the information Left of Right.

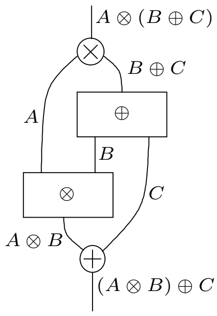
The graph is subject to two additional constraints, which essentially mean that

1. the incoming (resp. outgoing) arrows of a rule node uniquely match the hypothesis (resp. conclusions) of the corresponding rule of the sequent calculus;

2. each formula node has a unique incoming and at most one outgoing arrow.

Translating derivations into proof structures is straightforward. Not all proof structures are the images of derivations; those that are called *proof nets*. (When a graph is a proof net can also be characterized by the *switching criterion* introduced in (Danos & Regnier 1989), which requires that certain subgraphs of the proof structure be connected and acyclic.) Robinson's nets, with minor notational changes, were used in (Führmann & Pym 2004*b*).

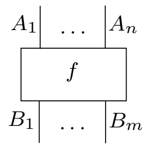
However, in this article, we have adopted the nets introduced in (Blute et al. 1996). In that style, the net for the derivations above



.

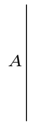
Here, the only nodes are rule nodes. We write $\otimes$ for $\wedge$, $\oplus$ for $\vee$, and $A^{\perp}$ for $\neg A$, because these nets are also used to describe morphisms in symmetric linearly distributive categories, as we shall see in § 2.4.2. The wires are labelled with *types*, which can be seen either as formulæ or as objects of a symmetric linearly distributive category. The left-hand formula of the derived sequent appears at the top of the net, and the right-hand formula at the bottom. The top-to-bottom orientation has advantages over the left-to-right orientation with respect to the alignment of types and wires. It also ensures the nice property that a net is planar if and only if the corresponding derivations are within non-commutative logic, that is, they contain no exchange rules, cf. (Blute et al. 1996). An important difference between nets in the style of Robinson and nets in the style of (Blute et al. 1996) is that the latter have no axiom links and no cut links. Abandoning these links is possible because a cut and an axiom cancel each other out according to a (poly)categorical neutrality law (Führmann & Pym 2004*b*). Another difference is that Robinson's nets have links for weakening and contraction, while nets in the style of (Blute et al. 1996) do not. (However, we shall see that such links can be easily added to the latter.) It is a bit harder to make the leap from derivations to nets in the style of (Blute et al. 1996) than to nets in the style of Robinson. However, the former are better for heavy calculations, because they have no cluttering cut links and axiom links, and because one can drop the type annotations when they are clear from the context. (Just as one sometimes omits type annotations of lambda terms). This is why we opted for them in this article.

Now we turn towards a formal definition of nets, based on the definition in (Blute et al. 1996), but not quite as formal. We define the notion of *typed circuit*. Building a typed circuit requires a set $\mathcal{T}$ of *types* and a set $\mathcal{C}$ of *components*. Each component $f \in \mathcal{C}$ has a list $\alpha = (A_1, \ldots, A_n)$ of types describing the *input* *ports*, and a list $\beta = (B_1, \ldots, B_m)$ of types describing the *output ports* of f.



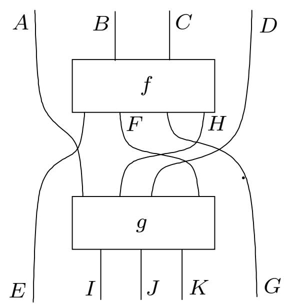
We define the collection of *circuits over* $\mathcal{C}$ inductively:

- Every component $f \in \mathcal{C}$ is a circuit.
- The *identity wire*



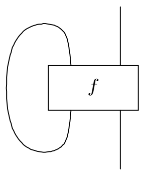
  is a circuit, with one input port and one output port, each of type A.
- Given any number of circuits, connecting some output ports with input ports of the same type yields another circuit.

For and example circuit, consider



Note that it has two connections (of types F and H) from f to g. As we shall see, this is *not* a net for symmetric linearly distributive categories, because those nets must have exactly one connection between any two components; however, the nets for symmetric monoidal categories that we shall introduce much later in § 3.4.1 allow such multiple connections.

Remark 2.2. Our definition of circuit is more general than that in (Blute et al. 1996) in that it allows *feedback*, e.g.



,

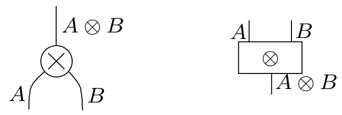
which we shall employ only in § 4.

A *net* (for symmetric linearly distributive categories), in short, is a circuit built from components that correspond to the introduction rules of the sequent calculus, subject to the condition of *sequentiality*, which means that the circuit must represent a derivation. We shall now spell this out in detail.

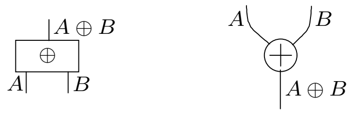
The types for nets are given by the grammar

$$A, B ::= A \otimes B \mid A \oplus B \mid \top \mid \bot \mid A^{\bot} \mid b,$$

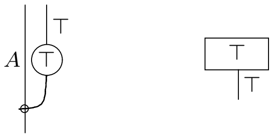
where b ranges over atomic formulæ. We have components

$\otimes L$  $\otimes R$

for conjunction, and components

$\oplus L$  $\oplus R$

for disjunction. We have components

$\top L$  $\top R$

to deal with $\top$.

Remark 2.3. A curiosity here is that $\top$L requires the *supporting wire A*. The wire that is directly attached to the supporting wire is called *thinning link* in (Blute et al. 1996). Thinning links are needed because of categorical coherence issues: for example, using nets without thinning links would force the identity morphism on $\top \oplus \top$ to be equal to the twist map, which is false in some symmetric linearly distributive categories (see (Blute et al. 1996)).

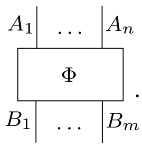
Dually, we have components

$\bot L$  $\bot R$

to deal with $\bot$. The components $\top$L and $\bot$R are called *thinning links*. When we consider negation, we also use components

¬L  ¬R.

Table 6 describes how a derivation Φ of $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is turned into a circuit



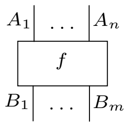
.

The double lines labelled Γ_i or Δ_j stand for bundles of wires, one for every formula contained in Γ_i or Δ_j. In the translations for $\top$L and $\bot$R, any wire in Φ can be used as a supporting wire. (We shall consider any two choices of supporting wire to be equivalent, see § 2.2.1 .)

We call the components $\otimes L$, $\otimes R$, $\oplus L$, $\oplus R$, $\top L$, $\top R$, $\bot L$, $\bot R$, $\neg$L, and $\neg$R *links* to distinguish them from arbitrary components. Links depicted by rectangular boxes correspond to axioms (e.g. Ax $\wedge$ R, Ax $\vee$ L, Ax$\neg$L, Ax$\neg$R, $\bot$L, $\top$R); they are nets. Links with circles correspond to inference rules which have one or more hypotheses; they are used to build nets, but they are not nets. (This is a notational clarification we adopt from (Cockett, Koslowski & Seely 2003).)

Definition 2.4. A *net over a signature* Σ is a circuit

- whose types are the formulæ over Σ,
- whose components are the links $\otimes L$, $\otimes R$, $\oplus L$, $\oplus R$, $\top L$, $\top R$, $\bot L$, $\bot R$, $\neg$L, $\neg$R, and components of the form

  ,

  where $\dfrac{}{A_1, \dots, A_n \vdash B_1, \dots, B_m} f$ is an optional axiom of Σ,
- which is in the image of translation in Table 6.

We call a net *positive* if all of its formulæ are positive. We write $\mathit{Net}(\Sigma)$ (resp. $\mathit{Net}^{\neg}(\Sigma)$) for the positive (resp. arbitrary) nets over Σ. We write $\mathit{Net}(\Sigma)(\Gamma, \Delta)$ (resp. $\mathit{Net}^{\neg}(\Sigma)(\Gamma, \Delta)$) for the positive (resp. arbitrary) nets with input ports according to Γ and output ports according to Δ.

2.2.1 Net-equivalence

In this section, we shall recall the equivalence between nets introduced in (Blute et al. 1996). It is defined by a number of rules for rewriting subcircuits. *These rules can only be applied if the both original circuit whose subcircuit is rewritten and the resulting circuit are nets.*

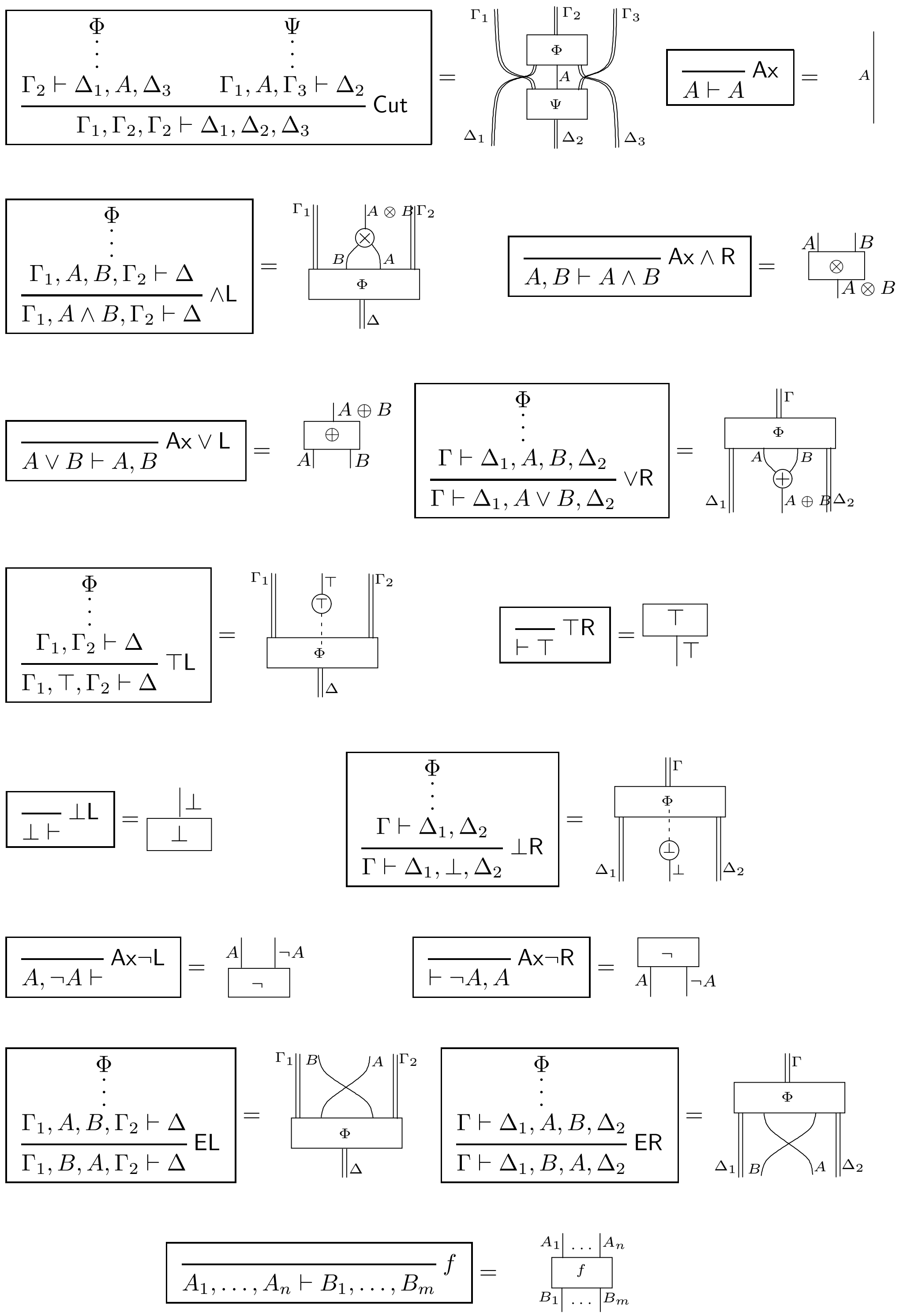
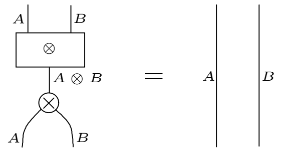
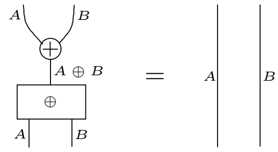
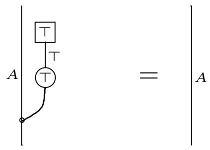
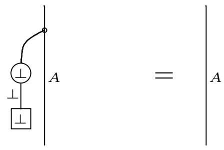
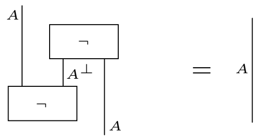


Table 6: From derivations to nets

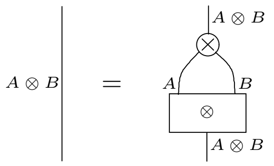
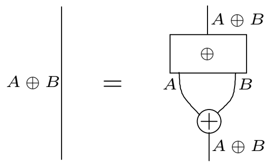
First, we have *reductions* that simulate the cut elimination of MLL:

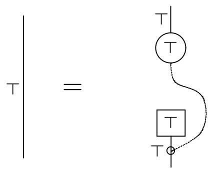
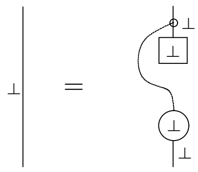
Reduce$\otimes$   *Reduce*$\oplus$

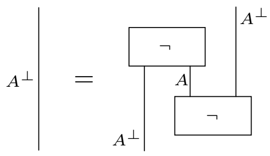
Reduce$\top$   *Reduce*$\bot$

 *Reduce*$\neg$.

Second, we have *expansions* that allow to express an axiom on a compound formula in terms of axioms on the subformulæ:

Expand$\otimes$   *Expand*$\oplus$

Expand$\top$   *Expand*$\bot$

 *Expand*$\neg$.

Finally, (Blute et al. 1996) contains a large number of rewriting rules that deal with the manipulation of thinning links. Fortunately, in the case of commutative logic, these rules amount to the *empire rewiring* proposition (Prop. 3.3 in (Blute et al. 1996)), which states that the supporting wire can be chosen freely within the empire[1] of the formula introduced by the thinning. This amounts to saying that the supporting wire can be chosen freely within *any* net containing the original supporting wire. For a detailed discussion of rewiring, see (Blute et al. 1996).

[1]The *empire* of a formula is the largest subnet containing that formula as an input port or an output port.

2.3 Symmetric linearly distributive categories

Linearly distributive categories, which are due to Cockett and Seely and which were initially called "weakly distributive categories", can be used to model MLL. (This is explained in (Cockett & Seely 1997*b*)—however, we shall spell out the semantics in § 2.4.1.) All logical systems we consider in this article are commutative—that is, they allow unrestricted use of the exchange rule; this allows us to use *symmetric* linearly distributive categories.

A symmetric linearly distributive category (Cockett & Seely 1997*b*) is a category $\mathbf{C}$ with two symmetric monoidal structures

$$\begin{array}{rr} \otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C} & \oplus : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C} \\ \top \in Ob(\mathbf{C}) & \bot \in Ob(\mathbf{C}) \\ \alpha_\otimes : (A \otimes B) \otimes C \cong A \otimes (B \otimes C) & \alpha_\oplus : (A \oplus B) \oplus C \cong A \oplus (B \oplus C) \\ \lambda_\otimes : \top \otimes A \cong A & \lambda_\oplus : \bot \oplus A \cong A \\ \rho_\otimes : A \otimes \top \cong A & \rho_\oplus : A \otimes \bot \cong A \\ \sigma_\otimes : A \otimes B \longrightarrow B \otimes A & \sigma_\oplus : A \oplus B \longrightarrow B \oplus A \end{array}$$

and a natural transformation

$$\delta : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$

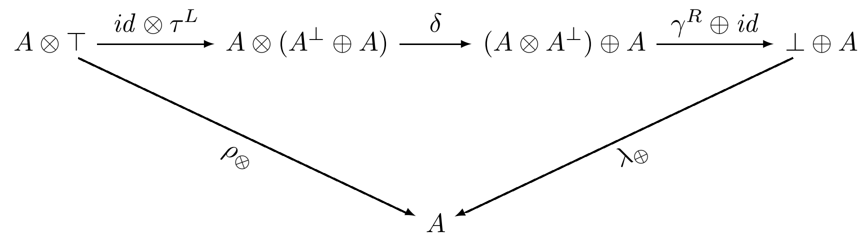
called *(linear) distribution*, which must satisfy various coherence conditions. For a description of those conditions, see (Cockett & Seely 1997*b*). The distribution is used to model the cut rule, as we shall explain in § 2.4.1.

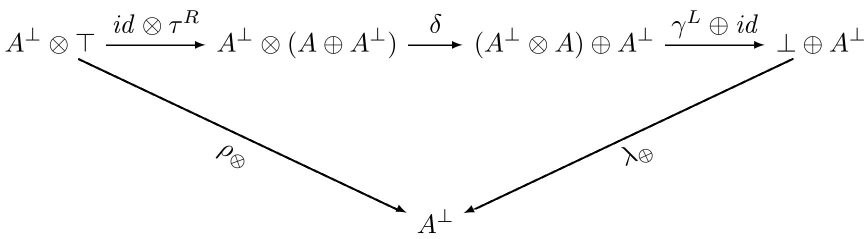
We call $\otimes$ the *tensor* and $\oplus$ the *cotensor*. (Not to be confused with the cotensor product of modules.)

A *symmetric linearly distributive category with negation* is a symmetric linearly distributive category together with, for every object A, an object $A^\perp$, and maps

$$\gamma^R : A \otimes A^\perp \longrightarrow \bot \qquad \tau^R : \top \longrightarrow A \oplus A^\perp$$

satisfying the conditions below (Cockett & Seely 1997*b*),





where γ^L and τ^L are the evident maps resulting from γ^R and τ^R by composing with symmetry maps. These maps can be used to model $\mathsf{Ax}\neg\mathsf{L}$ and $\mathsf{Ax}\neg\mathsf{R}$, as we shall explain in § 2.4.1.

Symmetric linearly distributive categories with negation are equivalent to $*$-autonomous categories (Cockett & Seely 1997*b*).

Finally, we recall a notion that plays an important rôle in the GoI construction: a *compact closed* category is a symmetric linearly distributive category $\mathbf{C}$ with negation such that the symmetric monoidal categories $(\mathbf{C}, \otimes, \top)$ and $(\mathbf{C}, \oplus, \bot)$ are identical, and δ is the associativity map.

Remark 2.5. Alternatively, one could define a compact closed category to be a symmetric monoidal category with, for every object A, an assigned left adjoint $A^\perp$ (Kelly & Laplaza 1980). The degenerate versions of the two equational laws for γ and τ are the triangular identities of that adjunction.

2.4 Categorical semantics of MLL

In this section, we recall the semantics of MLL in symmetric linearly distributive categories. In § 2.4.1, we describe the interpretation of derivations as morphisms. In § 2.4.2, we switch from derivations to nets, because nets allows smoother presentation. At the end of § 2.4.2, we state the important result that MLL nets (and therefore also derivations) are in perfect correspondence with symmetric linearly distributive categories (Theorem 2.6).

2.4.1 The interpretation of sequents

An *interpretation* for a signature Σ in a symmetric linearly distributive category $\mathbf{C}$ sends every formula A over Σ to an object $\lfloor A \rfloor$ according to the rules

$$\lfloor A \wedge B \rfloor = \lfloor A \rfloor \otimes \lfloor B \rfloor \qquad \lfloor \top \rfloor = \top$$
$$\lfloor B \vee B \rfloor = \lfloor A \rfloor \oplus \lfloor B \rfloor \qquad \lfloor \bot \rfloor = \bot.$$

If we consider the scenario with negation, then $\mathbf{C}$ must be a symmetric linearly distributive category with negation, and we also require

$$\lfloor \neg A \rfloor = \lfloor A \rfloor^\perp.$$

A derivation Φ of a sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is interpreted by a morphism

$$\left\lfloor \begin{array}{c} \Phi \\ \vdots \\ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \end{array} \right\rfloor : \lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor \longrightarrow \lfloor B_1 \rfloor \oplus \cdots \oplus \lfloor B_m \rfloor,$$

where $\otimes$ and $\oplus$ are, say, left associative, the tensor for $n = 0$ is $\top$, and the cotensor for $m = 0$ is $\bot$.

1. The rule Ax is interpreted by the identity morphism, and so are $\mathsf{Ax} \wedge \mathsf{R}$, $\mathsf{Ax} \vee \mathsf{L}$, $\mathsf{Ax}\top\mathsf{R}$, and $\mathsf{Ax}\bot\mathsf{L}$.

2. The rules $\wedge$L resp. $\top$L are interpreted by pre-composing the symmetric monoidal isomorphisms

$$\lfloor\Gamma\rfloor \otimes (\lfloor A\rfloor \otimes \lfloor B\rfloor) \otimes \lfloor\Gamma'\rfloor \cong \lfloor\Gamma\rfloor \otimes \lfloor A\rfloor \otimes \lfloor B\rfloor \otimes \lfloor\Gamma'\rfloor$$

resp.

$$\lfloor\Gamma\rfloor \otimes \top \otimes \lfloor\Gamma'\rfloor \cong \lfloor\Gamma\rfloor \otimes \lfloor\Gamma'\rfloor .$$

Dually for $\vee$R and $\bot$R.

3. The rule EL is interpreted by pre-composing the symmetric-monoidal isomorphism

$$\lfloor\Gamma_1\rfloor \otimes \lfloor A\rfloor \otimes \lfloor B\rfloor \otimes \lfloor\Gamma_2\rfloor \cong \lfloor\Gamma_1\rfloor \otimes \lfloor B\rfloor \otimes \lfloor A\rfloor \otimes \lfloor\Gamma_2\rfloor .$$

Dually for ER.

4. The cut rule is interpreted as follows: if the interpretations of the premises are

$$\left\lfloor \begin{array}{c} \Phi \\ \vdots \\ \Gamma_2 \vdash \Delta_1, A, \Delta_3 \end{array} \right\rfloor = f : \lfloor\Gamma_2\rfloor \longrightarrow \lfloor\Delta_1\rfloor \oplus \lfloor A\rfloor \oplus \lfloor\Delta_3\rfloor$$

$$\left\lfloor \begin{array}{c} \Psi \\ \vdots \\ \Gamma_1, A, \Gamma_3 \vdash \Delta_2 \end{array} \right\rfloor = g : \lfloor\Gamma_1\rfloor \otimes \lfloor A\rfloor \otimes \lfloor\Gamma_3\rfloor \longrightarrow \lfloor\Delta_2\rfloor ,$$

then the interpretation

$$\left\lfloor \dfrac{\begin{array}{c} \Phi \\ \vdots \\ \Gamma_2 \vdash \Delta_1, A, \Delta_3 \end{array} \qquad \begin{array}{c} \Psi \\ \vdots \\ \Gamma_1, A, \Gamma_3 \vdash \Delta_2 \end{array}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\,\mathsf{Cut} \right\rfloor$$

of the conclusion is

$$\begin{aligned} \lfloor\Gamma_1\rfloor \otimes \lfloor\Gamma_2\rfloor \otimes \lfloor\Gamma_3\rfloor &\xrightarrow{id\otimes f\otimes id} \lfloor\Gamma_1\rfloor \otimes (\lfloor\Delta_1\rfloor \oplus \lfloor A\rfloor \oplus \lfloor\Delta_3\rfloor) \otimes \lfloor\Gamma_3\rfloor \\ &\xrightarrow{\delta_1} \lfloor\Delta_1\rfloor \oplus (\lfloor\Gamma_1\rfloor \otimes \lfloor A\rfloor \otimes \lfloor\Gamma_3\rfloor) \oplus \lfloor\Delta_3\rfloor \\ &\xrightarrow{id\oplus g\oplus id} \lfloor\Delta_1\rfloor \oplus \lfloor\Delta_2\rfloor \oplus \lfloor\Delta_3\rfloor , \end{aligned}$$

where δ_1 is obtained by combining the distribution δ and structural isomorphisms of the symmetric monoidal category. (There are different such combinations, but the coherence laws of a symmetric linearly distributive category ensure that they all amount to the same morphism.)

5. If we consider the scenario with negation, Ax$\neg$L (resp. Ax$\neg$R) are interpreted by γ^L (resp. τ^R).

We shall describe the semantics of weakening and contraction later in this article.

Evidently, an interpretation of a derivation is determined by its action on the optional axioms.

2.4.2 Nets as symmetric linearly distributive categories

Our goal in this section is to explain the perfect correspondence between MLL and symmetric linearly distributive categories (with negation). To build a term model, we could construct a symmetric linearly distributive category whose morphisms are equivalence classes of derivations. However, the range of required equational laws would be almost unmanagable, because of countless commuting conversions and laws involving the exchange rule. Nets turn out to work much better here, because they deal with commuting conversions and exchange implicitly.

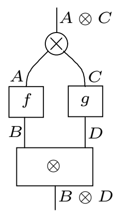
We believe that the transition from derivations to nets is harmless, because translating derivations into nets is almost trivial. (The nets can essentially be drawn into the derivation!)

The aim of this section is to describe how nets can be used to construct free symmetric linearly distributive categories (Theorems 2.6 and 2.8, which are taken from (Blute et al. 1996)).

Given a set E of equivalences on $Net(\Sigma)$ (where two nets can only be equivalent if they inhabit the same sequent), we can construct a symmetric linearly distributive category $Net_E(\Sigma)$ as follows:

- The objects are the formulæ over Σ.
- A morphism from A to B is a net $f \in Net(\Sigma)(A, B)$ modulo the congruence relation generated from E and the reductions, expansions, and empire rewiring equations described in § 2.2.1.
- Composition is defined in the evident way by connecting wires.
- The identity morphism on A is given by the wire labelled with A.
- Given nets representing morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$, the net representing $f \otimes g$ is defined as below. Dually for $\oplus$.

$f \otimes g =$



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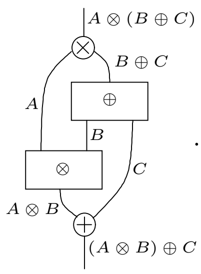
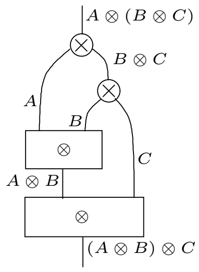
- The distribution is given by

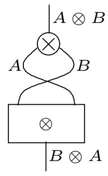
$\delta =$

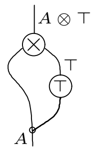


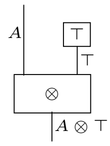
.

- The symmetric monoidal isomorphisms with respect to $\otimes$ are given by

$\alpha_\otimes =$ 

$\sigma_\otimes =$ 

$\rho_\otimes =$ 

$\rho_\otimes^{-1} =$  .

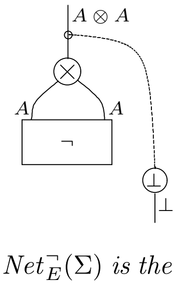
The remaining isomorphisms $\alpha_\otimes^{-1}$, $\lambda_\otimes$, and $\lambda_\otimes^{-1}$ are obvious. Dually for $\oplus$.

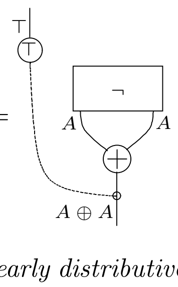
We have the following result from (Blute et al. 1996):

Theorem 2.6. *$Net_E(\Sigma)$ is the free symmetric linearly distributive category generated by the signature Σ and the equations E.*

Remark 2.7. This theorem implies soundness and completeness when $Net_E(\Sigma)$ is viewed as a theory whose judgments are equalities between nets. Completeness means that a judgment $M = N$ holds in the theory $Net_E(\Sigma)$ whenever it holds in every model; this is true because the theory $Net_E(\Sigma)$ forms a model of itself. Soundness means that every interpretation of the nets over Σ in a symmetric linearly distributive category **C** validates the equations in § 2.2.1. This is true because the canonical functor from $Net_E(\Sigma)$ to **C** is well-defined (i.e., it sends equivalent nets to the same morphism).

The free construction can be extended with negation. We only have to replace $Net(\Sigma)$ with $Net^\neg(\Sigma)$, allow the equations *Reduce*$\neg$ and *Expand*$\neg$, and define

$\gamma^L =$ 

$\tau^R =$  .

Theorem 2.8. *$Net_E^\neg(\Sigma)$ is the free symmetric linearly distributive category with negation generated by the signature Σ and the equations E.*

3 Modelling weakening and contraction: Dummett categories

In this section, we introduce categories that are in very close correspondence with the classical sequent calculus modulo cut-reduction. We proceed by extending the scenario for MLL presented in § 2.4 with structure for modelling weakening and contraction.

In § 3.1, we shall start with symmetric linearly distributive categories and add symmetric monoids and symmetric comonoids to model weakening and contraction. In particular, we shall present a remarkable result (explained to us by Hasegawa) that monoids or comonoids force symmetric linearly distributive categories to be MIX (Theorem 3.11).

In § 3.2, we shall add a poset-enrichment to model cut-reduction in the presence of weakening and contraction. We call the resulting categories *Dummett categories*. We do not require a Dummett category to have negation; if it has, we call it a *classical category*.

In § 3.3, we explore the structural properties of Dummett categories. In particular, we show that every hom-set of a Dummett category is a semilattice, in terms of which the poset-enrichment can be defined (Prop. 3.28). Moreover, we show that Dummett categories have an axiomatization in terms of unconditional equalities 3.31. Finally, we use this to show that the construction of the free symmetric linearly distributive category can be extended to Dummett categories and classical categories (Theorem 3.32).

In § 3.4, we study the important case of compact Dummett categories. (Our extended GoI construction later in the article involves only compact Dummett categories, and relies heavily on this section.) Compactness allows great simplifications of the nets and the axiomatization. In particular, we shall present an axiomatization of compact Dummett categories in terms of only one equality (Prop. 3.39). Moreover, we shall show how compact Dummett categories shed light on cut-reductions involving contraction (Prop. 3.38).

Finally, we specialize the compact setting to categories with finite biproducts, explain the resulting matrix calculus, and present a single equation that characterizes when a category with finite biproducts is a Dummett category (Prop. 3.40).

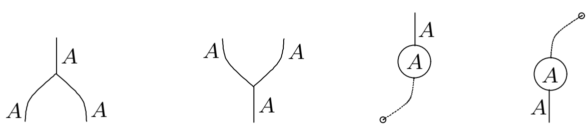
3.1 Symmetric monoids and comonoids

To model AxCL, AxCR, AxWL, and AxWR in a symmetric linearly distributive category, we introduce maps

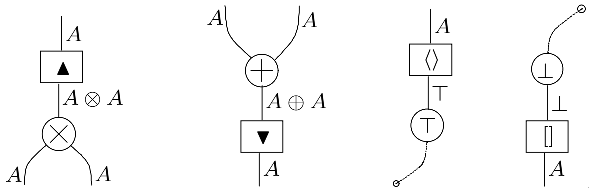
$$\begin{array}{ll} \blacktriangle_A : A \longrightarrow A \otimes A & \qquad \blacktriangledown_A : A \oplus A \longrightarrow A \\ \langle\rangle_A : A \longrightarrow \top & \qquad []_A : \bot \longrightarrow A \end{array}$$

for every object A.

Definition 3.1. When we use nets, we shall use the abbreviations

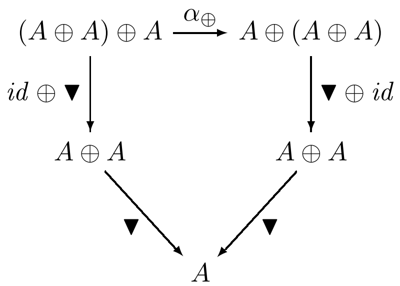


for

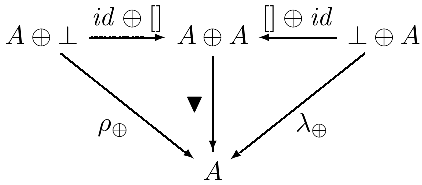


.

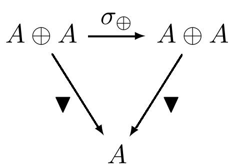
We shall require certain conditions to ensure that ▼, [], ▲, and ⟨⟩ are sensibly defined: we require $(A, \blacktriangledown_A, []_A)$ to be a symmetric monoid—that is, the associativity, neutrality, and commutativity laws below have to hold.



$\blacktriangledown assoc$



$[]neutral$



$\blacktriangledown symm\,.$

As is easy to see, these laws correspond to the following widely-accepted equalities between sequent proofs:

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots \\ \Gamma \vdash \Delta, A, A, A, \Delta'\end{matrix}}{\textsf{CR}\text{ applied to the left } A \text{ and the middle } A}}{\dfrac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\,\textsf{CR}} = \dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots \\ \Gamma \vdash \Delta, A, A, A, \Delta'\end{matrix}}{\textsf{CR}\text{ applied to the middle } A \text{ and the right } A}}{\dfrac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\,\textsf{CR}}$$

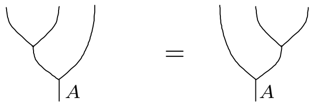
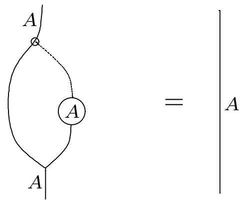
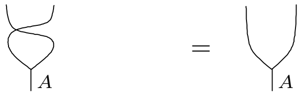
	CR-*assoc*
	WR-*neutral*
	CR-*symm*

Table 7: Net-equalities for symmetric monoids

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots \\ \Gamma \vdash \Delta, A, \Delta'\end{matrix}}{\Gamma \vdash \Delta, A, A, \Delta'}\ \text{WR applied in either of the two evident ways}}{\Gamma \vdash \Delta, A, \Delta'}\ \mathsf{CR} \;=\; \begin{matrix}\Phi\\ \vdots \\ \Gamma \vdash \Delta, A, \Delta'\end{matrix}$$

$$\dfrac{\begin{matrix}\Phi\\ \vdots \\ \Gamma \vdash \Delta, A, A, \Delta'\end{matrix}}{\Gamma \vdash \Delta, A, A, \Delta'}\ \mathsf{ER} \;=\; \begin{matrix}\Phi\\ \vdots \\ \Gamma \vdash \Delta, A, A, \Delta'.\end{matrix}$$

The net-versions of these laws are presented in Table 7. (Owing to symmetry, we need only one of the two neutrality laws.) Moreover, we require for all objects A and B that the monoid on $A \oplus B$ is defined pointwise in terms of the monoids on A and B; that is, we require

$$\begin{array}{ccc} A \oplus B \oplus A \oplus B & \xrightarrow{id \oplus \sigma_{\oplus} \oplus id} & A \oplus A \oplus B \oplus B \\ & {\scriptstyle \blacktriangledown_{A \oplus B}} \searrow \quad \swarrow {\scriptstyle \blacktriangledown_A \oplus \blacktriangledown_B} & \\ & A \oplus B & \end{array} \qquad \blacktriangledown \textit{pointwise},$$

$$\begin{array}{ccc} \bot & \cong & \bot \oplus \bot \\ & {\scriptstyle []_{A\oplus B}} \searrow \quad \swarrow {\scriptstyle []_A \oplus []_B} & \\ & A \oplus B & \end{array} \qquad\qquad []\mathit{pointwise},$$

and also the nullary cases

$$\begin{aligned} &\blacktriangledown_\bot = \lambda_\oplus : \bot \oplus \bot \longrightarrow \bot \\ &[]_\bot = id_\bot \qquad\qquad\qquad\qquad\qquad []\mathit{trivial}. \end{aligned}$$

As can be easily checked, the two nullary laws are interderivable; in the remainder of this article, we shall stick with $[]\mathit{trivial}$ and mention the other law no more. The laws $\blacktriangledown\mathit{pointwise}$, $[]\mathit{pointwise}$, and $[]\mathit{trivial}$ correspond to the following equalities between sequent proofs:

$$\dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, B, A, B, \Delta'\end{array}}{\Gamma \vdash \Delta, A \wedge B, A \wedge B, \Delta'}\text{ two applications of } \wedge\mathsf{R}}{\Gamma \vdash \Delta, A \wedge B, \Delta'}\,\mathsf{CR} \;=\; \dfrac{\dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, B, A, B, \Delta'\end{array}}{\Gamma \vdash \Delta, A, A, B, B, \Delta'}\,\mathsf{ER}}{\Gamma \vdash \Delta, A, B, \Delta'}\text{ two applications of } \mathsf{CR}}{\Gamma \vdash \Delta, A \wedge B, \Delta'}\,\wedge\mathsf{R}$$

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, \Delta'\end{array}}{\Gamma \vdash \Delta, A \wedge B, \Delta'}\,\mathsf{WR} \;=\; \dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, \Delta'\end{array}}{\Gamma \vdash \Delta, A, B, \Delta'}\text{ two applications of } \mathsf{WR}}{\Gamma \vdash \Delta, A \wedge B, \Delta'}\,\mathsf{WR}$$

$$\dfrac{\bot \vdash}{\bot \vdash \bot}\,\mathsf{WR} = \dfrac{}{\bot \vdash \bot}\,\mathsf{Ax}.$$

The net-versions of these laws are presented in Table 8.

Remark 3.2. While we believe that the laws in Table 7 are hard to dismiss (logicians seem to use them implicitly), the laws in Table 8 are perhaps more contentious. We require them because they seem highly plausible and required for numerous propositions and constructions.

Dually, we shall use comonoids ($\blacktriangle_A : A \longrightarrow A \otimes A, \langle\rangle_A : A \longrightarrow \top$) to model left contraction and weakening. The laws for comonoids are called $\blacktriangle\mathit{assoc}$, $[]\mathit{neutral}$, $\blacktriangle\mathit{symm}$, $\blacktriangle\mathit{pointwise}$, $\langle\rangle\mathit{pointwise}$, and $\langle\rangle\mathit{trivial}$. Their net-versions are called CL-*assoc*, WL-*neutral*, CL-*symm*, CL-*pointwise*, WL-*pointwise*, and WL-*trivial*.

Definition 3.3. A symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \oplus, \bot)$ is said to have *symmetric monoids* if every object A has a chosen symmetric monoid $(\blacktriangledown_A, []_A)$, and the laws $\blacktriangledown\mathit{pointwise}$, $[]\mathit{pointwise}$, and $[]\mathit{trivial}$ hold.

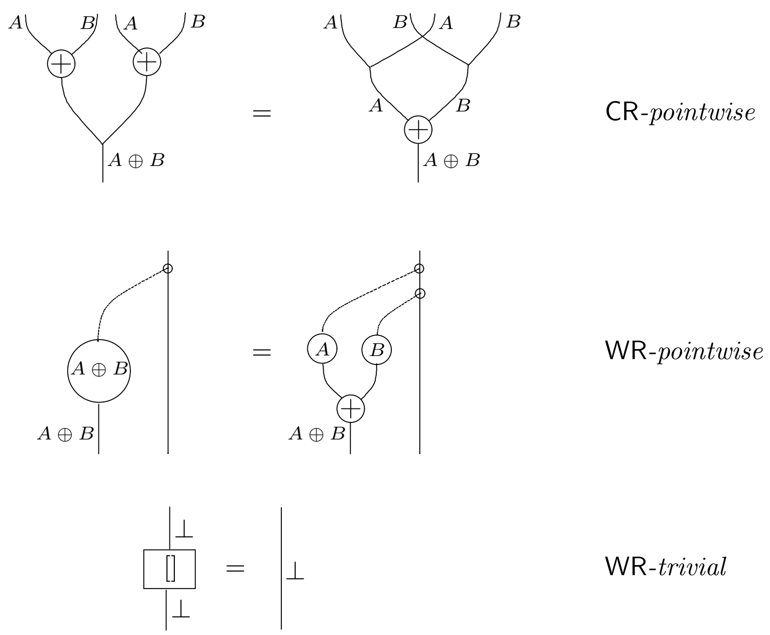


Table 8: Net-equalities for the pointwise definition of the symmetric monoids

Dually, a symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, \top)$ is said to have *symmetric comonoids* if every object A has a chosen symmetric comonoid $(\blacktriangle_A, \langle\rangle_A)$, and the laws $\blacktriangle$*pointwise*, $\langle\rangle$*pointwise*, and $\langle\rangle$*trivial* hold.

Definition 3.4. A *pre-Dummett category* is a symmetric linearly distributive category $\mathbf{C}$ such that

1. the symmetric monoidal category $(\mathbf{C}, \oplus, \bot)$ has symmetric monoids;
2. the symmetric monoidal category $(\mathbf{C}, \otimes, \top)$ has symmetric comonoids.

Remark 3.5. This agrees with Hasegawa's notion of pre-Dummett category, except that we do not require the hom-semigroups (defined in § 3.3) be idempotent.

Example 3.6. Every distributive lattice $\mathbf{D}$. The objects are the elements of $\mathbf{D}$, and there is at most one morphism $A \longrightarrow B$, which exists if and only if $A \leq B$. The functor $\otimes$ is the greatest lower bound, and $\oplus$ is the least upper bound. The object $\top$ is the greatest element, and $\bot$ is the least element. The distribution exists because $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C) \leq (A \otimes B) \oplus C$. The monoids and comonoids exist because $A = A \otimes A$, $A \leq \top$, $A \oplus A = A$, and $\bot \leq A$ for all $A \in \mathbf{D}$.

Example 3.7. Every symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \odot, I)$ with symmetric monoids and symmetric comonoids, if both $\otimes$ and $\oplus$ are defined to be $\odot$, and both $\top$ and $\bot$ are defined to be I. The distribution is the associativity $A \odot (B \odot C) \cong (A \odot B) \odot C$. Examples of such categories include:

- The category **Rel**, whose objects are (small) sets, and whose morphism $A \longrightarrow B$ are subsets of $A \times B$, if $\odot$ is defined to be the evident functor that sends two sets to their set-theoretic product, and I is defined to be the singleton set $\{*\}$. We have $\blacktriangledown_A = \{((x, x), x) : x \in A\}$ and $[]_A = \{(*, x) : x \in A\}$. Dually for $\blacktriangle_A$ and $\langle\rangle_A$. We write $(\mathbf{Rel}, \times)$ for this pre-Dummett category.
- Every category with finite biproducts, if $\odot$ is defined to be the binary biproduct, and I is defined to be the zero (i.e., initial and terminal) object. The comonoids are given by the diagonals and projections of the product structure, and dually for the monoids. Examples include:
  - The category **Rel**, if $\odot$ is defined to be evident functor that sends two sets to their disjoint union, and I is defined to be the empty set. We write $(\mathbf{Rel}, \uplus)$ for this pre-Dummett category;
  - The category $\mathbf{FDVec}_K$ of finite-dimensional vector spaces over a field K, if $\odot$ is defined to be the "direct sum", which sends two spaces to their set-theoretic product, and I is defined to be the one-dimensional space K. We write $(\mathbf{FDVec}_K, \times)$ to distinguish it from the compact closed category based on the tensor product.

The product $\mathbf{C}_1 \times \mathbf{C}_2$ of two pre-Dummett categories is a pre-Dummett category. Letting $\mathbf{C}_1$ be a distributive lattice and $\mathbf{C}_2$ be any of the categories in Example 3.7 shows that there exist pre-Dummett categories with non-trivial hom-sets such that $\otimes \neq \oplus$ and $\top \neq \bot$.

Theorem 3.8. *Let Σ be a signature containing* AxWL, AxWR, AxCL, *and* AxCR. *Let E be a set of equations on $Net(\Sigma)$, and let E' be the set of equations for pre-Dummett categories described in Tables 7 and 8 and their duals. Then $Net_{E\cup E'}(\Sigma)$ is the free pre-Dummett category generated by Σ and E.*

Proof. This follows from Theorem 2.6, and the fact that E' characterizes pre-Dummett categories. $\square$

We finish this section with some definitions. In a pre-Dummett category, the morphisms π_1^{AB} and π_2^{AB} are defined to be

$$A \otimes B \xrightarrow{id\otimes\langle\rangle} A \otimes \top \cong A \qquad \text{and} \qquad A \otimes B \xrightarrow{\langle\rangle\otimes id} \top \otimes B \cong B,$$

respectively. Dually, ι_1^{AB} and ι_2^{AB} are defined to be the evident morphisms $A \longrightarrow A \oplus B$ and $B \longrightarrow A \oplus B$.

3.1.1 MIX

By *MIX rule*, we mean the following inference rule:

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{MIX}.$$

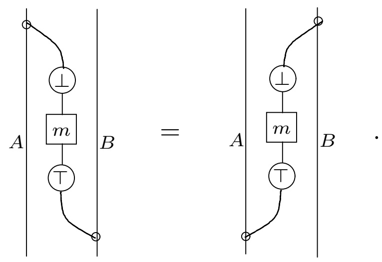
(This is the two-sided version of the MIX rule presented in (Girard 1987), not the MIX rule presented in (Gentzen 1934).) It is obviously derivable in the classical sequent calculus, for example as follows:

$$\frac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\,\mathsf{WR} \qquad \dfrac{\Gamma' \vdash \Delta'}{\bot, \Gamma' \vdash \Delta'}\,\mathsf{WL}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{Cut}.$$

A (symmetric) linearly distributive category is called a *(symmetric) MIX category* or said to be *MIX* if it satisfies a certain condition (which we shall present below) that ensures that the MIX rule has a canonical semantics.

In this section, we show that every pre-Dummett category is MIX. In fact, we show a stronger result stating that every symmetric linearly distributive category with a monoid on $\bot$ or a comonoid on $\top$ is MIX (Theorem 3.11). The MIX property is interesting from a proof-theoretic point of view; it is also important for the equational characterization of Dummett categories (§ 3.3) and for our extended GoI construction (§ 4).

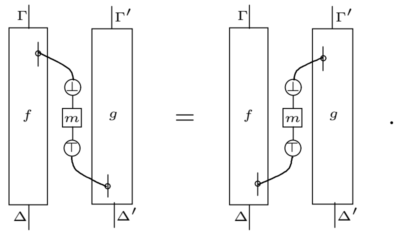
A (symmetric) *MIX category* is a (symmetric) linearly distributive category with a morphism $m : \bot \longrightarrow \top$ such that, for all objects A and B, the two evident morphisms $A \otimes B \longrightarrow A \oplus B$ agree (Cockett & Seely 1997*a*):



We write mix_{AB} for the canonical morphism $A \otimes B \longrightarrow A \oplus B$. The family mix_{AB} of morphisms is easily seen to be a natural in A and B.

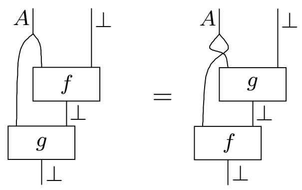
In a symmetric MIX category, the *MIX barbell*

provides a canonical way of gluing together any two nets f and g:



(The supporting wire of the thinning link within each net does not matter owing to the empire rewiring proposition mentioned in § 2.2.1.) So a symmetric MIX category provides a canonical semantics to the MIX rule.

Lemma 3.9. *Let* $\mathbf{C}$ *be a symmetric linearly category* $\mathbf{C}$ *with a morphism* $\blacktriangle_A : A \longrightarrow A \otimes A$. *Then for all* $f, g : A \otimes \bot \longrightarrow \bot$, *we have*



(1)

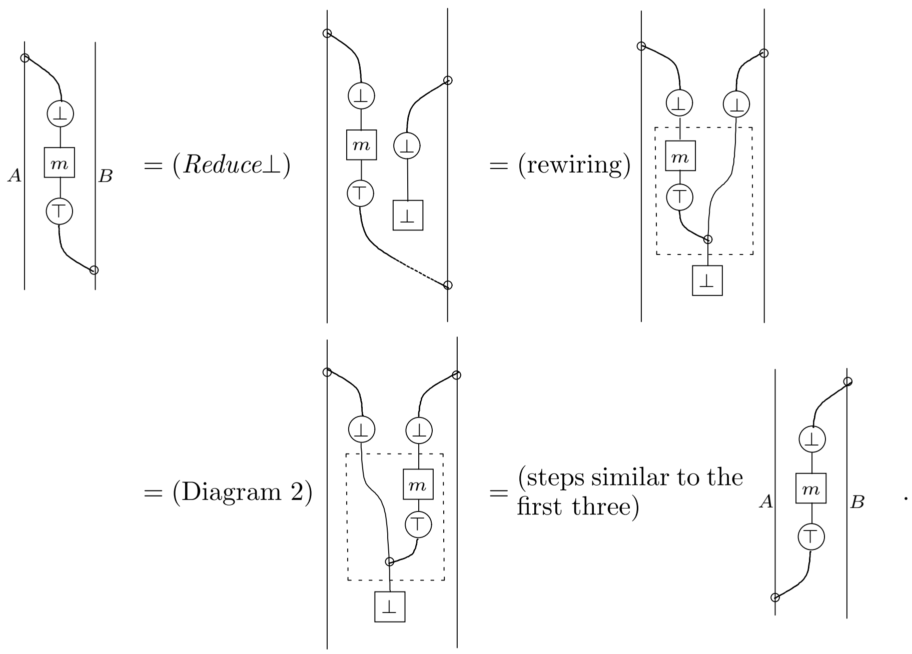
Similarly when each side of Equation 1 has $n = 0$ *or* $n \geq 2$ *copies of* A *as input (i.e., each side has* $n = 0$ *or* $n \geq 2$ *occurrences of* $\blacktriangle_A$*).*

Proof. See the Appendix. □

Lemma 3.10. *A linearly distributive category with a morphism* $m : \bot \longrightarrow \top$ *is MIX if and only if the diagram below commutes.*

$$\begin{array}{ccc} \bot \otimes \bot & \xrightarrow{m \otimes id} & \top \otimes \bot \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle \cong} \\ \bot \otimes \top & \xrightarrow{\cong} & \bot \end{array} \qquad (2)$$

Proof. For the right-to-left direction, suppose that Diagram 2 commutes. Then



The left-to-right direction, which plays no rôle in this article, follows from simple calculations; we leave the details to the reader. ☐

The following theorem was found by Hasegawa (private communications), except that we managed to remove the requirement that the comonoid (resp. monoid) be symmetric.

Theorem 3.11. *Every symmetric linearly distributive category with a comonoid*

$$\blacktriangle_\bot : \bot \longrightarrow \bot \otimes \bot \qquad \langle\rangle_\bot : \bot \longrightarrow \top$$

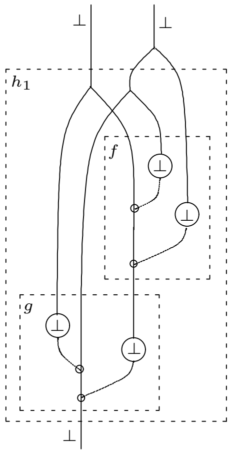
is MIX (with $m = \langle\rangle_\bot$). Dually , every symmetric linearly distributive category with a monoid

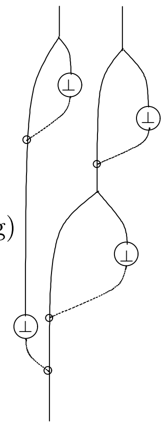
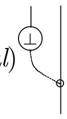
$$\blacktriangledown_\top : \top \oplus \top \longrightarrow \top \qquad []_\top : \bot \longrightarrow \top$$

is MIX (with $m = []_\top$).

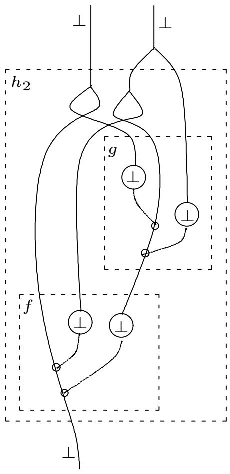
Proof. We show the comonoid case, with the help of Lemma 3.10: first, we present a net k_1 which denotes the top-right leg of Diagram 2 (with $m = \langle\rangle_\bot$), and another net k_2 which denotes the left-bottom leg; then, we use Lemma 3.9 to show that k_1 and k_2 are equal. The dashed boxes labelled f, g, h_1, and h_2

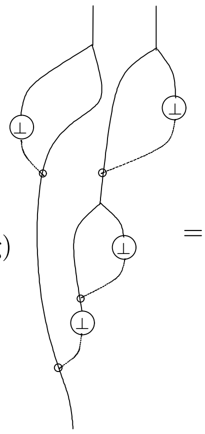
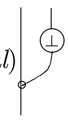
denote subnets.

$k_1 =$ 

$=$ (empire rewiring)  $=$ (WL-*neutral*) 

$$= \bot \otimes \bot \xrightarrow{\langle\rangle_\bot \otimes id} \top \otimes \bot \cong \bot.$$

$k_2 =$ 

$=$ (empire rewiring)  $=$ (WL-*neutral*) 

$$= \bot \otimes \bot \xrightarrow{id \otimes \langle\rangle_\bot} \bot \otimes \top \cong \bot.$$

By Lemma 3.9 with $n = 2$, we have $h_1 = h_2$, and therefore $k_1 = k_2$. $\square$

Corollary 3.12. *Every pre-Dummett category is MIX with $m = \langle\rangle_\bot$, and also with $m = []_\top$.*

We write $mix^{\langle\rangle}_{AB}$ (resp. $mix^{[]}_{AB}$) for the natural transformation $A \otimes B \longrightarrow A \oplus B$ built from $\langle\rangle_\bot$ (resp. $[]_\top$).

Are pre-Dummett categories canonically MIX—that is, do we have $\langle\rangle_\bot = []_\top$? We do not know the general answer to this question, but we shall see (Lemma 3.23) that the answer for Dummett categories is "yes".

3.2 Poset-enrichment

Our next goal is to model the cut-reduction rules for weakening and contraction—that is, the equations in Table 9 and their duals *Reduce*WR and

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma\vdash\Delta,A\end{array} \qquad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma'\vdash\Delta'\end{array}}{A,\Gamma'\vdash\Delta'}\,\mathsf{WL}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{Cut} \quad\leq\quad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma'\vdash\Delta'\end{array}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{WL,WR} \qquad \mathit{Reduce}\mathsf{WL}$$

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma\vdash\Delta,A\end{array} \qquad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ A,A,\Gamma'\vdash\Delta'\end{array}}{A,\Gamma'\vdash\Delta'}\,\mathsf{CL}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{Cut} \quad\leq\quad \dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma\vdash\Delta,A\end{array} \qquad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma\vdash\Delta,A\end{array} \qquad \begin{array}{c}\Psi\\ \vdots\\ A,A,\Gamma'\vdash\Delta'\end{array}}{A,\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{Cut}}{\Gamma,\Gamma,\Gamma'\vdash\Delta,\Delta,\Delta'}\,\mathsf{Cut}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{CL,CR} \qquad \mathit{Reduce}\mathsf{CL}$$

Table 9: "Cut-reductions for weakening and contraction (representative cases)"

*Reduce*CR. We use the symbol $\leq$ instead of the equality symbol, because we shall not require that the denotation of redex and reduct be the same: if we required them to be the same in the rules *Reduce*WL and *Reduce*WR, then any two derivations of $\Gamma \vdash \Delta$ would have the same denotation because of Lafont's example; if we required them to be the same in the rules *Reduce*CL and *Reduce*CR, we would rule out desirable models, as we shall see in Example 3.15.

Table 10 contains the net-versions of the reductions in Table 9. The derivation Φ corresponds to the net f. The net corresponding to Ψ is not needed, because we allow ourselves to rewrite subcircuits. We assume without loss of generality that Γ and Δ consist of single formulæ; this is possible because we can always bundle a wire labelled with $\Gamma = A_1, \ldots, A_n$ in a single wire labelled with $A_1 \otimes \cdots \otimes A_n$ (by using the two kinds of links for $\otimes$), and a wire labelled with $\Delta = B_1, \ldots, B_m$ in a single wire labelled with $B_1 \oplus \cdots \oplus B_m$ (by using the two kinds of links for $\oplus$).

In our categorical models, $\leq$ will be a poset-enrichment. Consider the net-version of the law *Reduce*CL; if Δ is empty, it corresponds to categorical law

$$\blacktriangle_A \circ f \leq (f \otimes f) \circ \blacktriangle_\Gamma \qquad\qquad \blacktriangle\mathit{lax},$$

which states that $\blacktriangle$ is a lax natural transformation. Similarly, the law *Reduce*WL for empty Δ corresponds to the categorical law

$$\langle\rangle_A \circ f \leq \langle\rangle_\Gamma \qquad\qquad \langle\rangle\mathit{lax},$$

which states that $\langle\rangle$ be a lax natural transformation.

The parametric categorical laws—that is, the versions for non-empty Δ, are very cumbersome: stating them requires multiple uses of the distribution δ; alternatively, one can stick with the non-parametric versions and add four extra inequalities (see Table 1 in (Führmann & Pym 2004*a*)). By contrast, the net-versions of the laws are elegant; moreover, equations between nets are perfectly

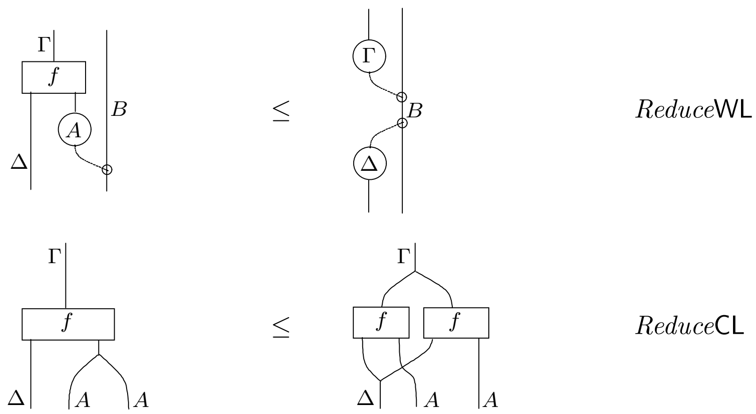


Table 10: "Net-version of Table 9"

suitable to describe equalities between morphisms, owing to Theorem 2.6. So we stick with the net-versions in this article.

Definition 3.13. A *Dummett category* is a pre-Dummett category **C** together with a poset-enrichment $\leq$ such that

1. the functors $\otimes$ and $\oplus$ are monotonic in both arguments;

2. the laws *Reduce*WL, *Reduce*WR, *Reduce*CL, and *Reduce*CR hold.

Example 3.14. Every distributive lattice **D** (which we know to be a pre-Dummett category from Example 3.6). The partial order is trivial, because each hom-set contain at most one element.

Example 3.15. The pre-Dummett category $(\mathbf{Rel}, \times)$, where for relations $f, f' : A \longrightarrow B$, we define $f \leq f' \iff f \subseteq f'$, where $\subseteq$ is the set-theoretic inclusion. To see that *Reduce*WL holds, let Γ, Δ, A, B be sets, and let f be a relation $\Gamma \longrightarrow \Delta \times A$. The relation denoted by the redex turns out to be

$$\{((g,b),(d,b)) : b \in B \wedge \exists a \in A : (g,(d,a)) \in f\},$$

while the reduct turns out to be

$$\{((g,b),(d,b)) : b \in B \wedge g \in \Gamma \wedge d \in \Delta\}.$$

The two are equal if and only if for all $g \in \Gamma$ and $d \in \Delta$, there exists an $a \in A$ such that $(g,(d,a)) \in f$. We call such relations $f : \Gamma \longrightarrow \Delta \times A$ *total*; for empty Δ, this agrees with the usual notion of total relation. Dually for *Reduce*WR. To see that *Reduce*CL holds, note that reduct turns out to be

$$\{(g,(d,a,a)) : (g,(d,a)) \in f\},$$

while the reduct turns out to be

$$\{(g,(d,a_1,a_2)) : (g,(d,a_1)) \in f \wedge (g,(d,a_2)) \in f\}.$$

The two are equal if and only if for all $g \in \Gamma$, $d \in \Delta$, and $a_1, a_2 \in A$, we have $a_1 = a_2$ whenever $(g,(d,a_1)) \in f$ and $(g,(d,a_2)) \in f$. We call such relations $f : \Gamma \longrightarrow \Delta \times A$ *functional*; for empty Δ, this agrees with the usual notion of functional relation.

Example 3.16. The pre-Dummett category $(\mathbf{Rel}, \uplus)$, where for relations $f, f' : A \longrightarrow B$, we define $f \leq f' \iff f' \subseteq f$. To see that *Reduce*WL holds, let Γ, Δ, A, B be sets, and let f be a relation $\Gamma \longrightarrow \Delta \uplus A$. Then f consists of components $f_{\Gamma\Delta} : \Gamma \longrightarrow \Delta$ and $f_{\Gamma A} : \Gamma \longrightarrow A$. The relation denoted by the redex, when presented as a 2×2-matrix, turns out to be

$$\left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & f_{\Gamma\Delta} & \emptyset \\ A & \emptyset & id_B \end{array}\right),$$

while the reduct is

$$\left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & \emptyset & \emptyset \\ A & \emptyset & id_B \end{array}\right),$$

The two are equal if and only if $f_{\Gamma\Delta} = 0$. Dually for *Reduce*WR. To see that *Reduce*CL holds, note that both redex and reduct turn out to be given by the 3×1-matrix

$$\left(\begin{array}{c|c} & \Gamma \\ \hline \Delta & f_{\Gamma\Delta} \\ A & f_{\Gamma A} \\ A & f_{\Gamma A} \end{array}\right).$$

Dually for *Reduce*CR.

As we shall see in Remark 3.26, the pre-Dummett category $(\mathbf{FDVec}_K, \times)$ of finite-dimensional vector spaces over a field K does not form a Dummett category.

As in the case of pre-Dummett categories, the product of two Dummett categories forms a Dummett category. The product of a distributive lattice and $(\mathbf{Rel}, \uplus)$ or $(\mathbf{Rel}, \times)$ shows that there are Dummett categories with non-trivial hom-sets such that $\otimes \neq \oplus$ and $\top \neq \bot$.

Definition 3.17. A *classical category* is a Dummett category with negation (in the sense of § 2.3).

Example 3.18. Every boolean lattice, where $A^\perp$ is the complement of A.

Example 3.19. The Dummett category $(\mathbf{Rel}, \times)$, with $A^\perp = A$. The maps γ^L and γ^R are $\{((a,a),*) : a \in A\}$; similarly for τ^L and τ^R.

The Dummett category $(\mathbf{Rel}, \uplus)$ does not have negation: because both $\bot$ and $\top$ are the empty set, the maps τ^L, τ^R, γ^L, and γ^R could only be the empty relations; so they could not satisfy the required equations. However, we shall see later that every traced Dummett category (e.g. $(\mathbf{Rel}, \uplus)$) induces a classical category via an extended Geometry of Interaction construction (Theorem 4.4).

As in the case of Dummett categories, the product of two classical categories forms a classical category. The product of a boolean lattice and $(\mathbf{Rel}, \times)$ shows that there are classical categories with non-trivial hom-sets such that $\otimes \neq \oplus$ and $\top \neq \bot$.

3.2.1 Homomorphisms

Next, we introduce certain kinds of homomorphisms for studying how morphisms of a Dummett category behave with respect to *Reduce*WL, *Reduce*WR, *Reduce*CL, and *Reduce*CR.

If the law

$$f \circ \blacktriangledown \leq \blacktriangledown \circ (f \oplus f) \qquad \blacktriangledown lax$$

holds for f as an *equality*, then f is a semigroup homomorphism. If the law

$$f \circ [] \leq [] \qquad []lax$$

holds for f as an equality, then f preserves the unit $[]$ of the monoid; in this case, we call f a *pointed homomorphism.* If both laws hold for f as equalities, then f is a monoid homomorphism. Dually, we have notions of cosemigroup homomorphism, copointed homomorphism, and comonoid homomorphism. Now we generalize this to the parametric case:

Definition 3.20. A morphism $f : \Gamma \longrightarrow \Delta \oplus A$ of a Dummett category is called

- *parametrized copointed homomorphism* if *Reduce*WL holds for f as an equality;
- *parametrized cosemigroup homomorphism* if *Reduce*CL holds for f as an equality;
- *parametrized comonoid homomorphism* if f is both of the above.

Dually for parametrized pointed/semigroup/monoid homomorphisms.

According to the discussions in Examples 3.15 and 3.16, the situation for $(\mathbf{Rel}, \times)$ and $(\mathbf{Rel}, \uplus)$ is as follows:

property	$(\mathbf{Rel}, \times)$	$(\mathbf{Rel}, \uplus)$
$f : \Gamma \longrightarrow \Delta \oplus A$ is a parametrized copointed homomorphism	if f is total	if $f_{\Gamma\Delta} = \emptyset$
$f : \Gamma \longrightarrow \Delta \oplus A$ is a parametrized cosemigroup homomorphism	if f is functional	always
$f : \Gamma \longrightarrow A$ is a copointed homomorphism	if f is total	always
$f : \Gamma \longrightarrow A$ is a cosemigroup homomorphism	if f is functional	always
$f : \Gamma \longrightarrow A$ is a comonoid homomorphism	if f is a total function	always

.

Dually for semigroup/pointed/monoid homomorphisms.

What keeps $(\mathbf{Rel}, \uplus)$ from identifying both reducts in Lafont's example is that not every morphism is a *parametrized* (co)pointed homomorphism!

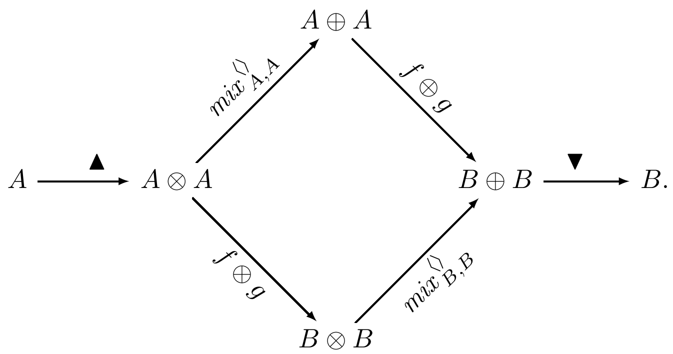
As we shall see in § 3.4.3, the homomorphism analysis for any Dummett category with finite biproducts leads to the same result that we found for $(\mathbf{Rel}, \uplus)$.

3.3 The structure of Dummett categories

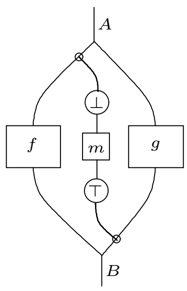
In this section, we show that the partial order of a Dummett category can be expressed in terms of the underlying pre-Dummett category (Prop. 3.28), and we use that result to show that Dummett categories can be axiomatized in terms of unconditional equations (Theorem 3.31). After that, we shall present the construction of the free Dummett category from nets (Theorem 3.32).

The key to the equational axiomatization is the observation that every hom-set of a pre-Dummett category has a binary operation $*$, which in the case of a Dummett category is a semilattice (not generally with a neutral element), from which the partial order $\leq$ can be derived.

Definition 3.21. For two morphisms $f, g : A \longrightarrow B$ of a pre-Dummett category, the morphism $f * g : A \longrightarrow B$ is defined as follows:



The corresponding net is



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where $m = \langle\rangle_\top$. That is, one glues f and g together with a MIX barbell (as discussed in § 3.1.1), obtaining a morphism $A \otimes A \longrightarrow B \oplus B$, and then one pre-composes $\blacktriangle$ and post-composes $\blacktriangledown$. (Re-attaching the bottom thinning link of the MIX barbell to the wire above g would yield the upper leg $A \longrightarrow B$ of the commuting diagram above, while re-attaching the top thinning link of the MIX barbell to the wire below f would yield the bottom leg $A \longrightarrow B$ of the commuting diagram above. Both nets are equal to the net above owing to empire rewiring.)

We shall soon see (Lemma 3.23) that $\langle\rangle_\bot = []_\top$ in the case of a Dummett category; so in that case it does not matter whether we use $mix^{\langle\rangle}$ or $mix^{[]}$ in the definition of the operation $*$.

Example 3.22. In $(\mathbf{Rel}, \times)$, where $mix_{AB} = id_{A\times B}$, $*$ is the set-theoretic intersection. In $(\mathbf{Rel}, \uplus)$, where $mix_{AB} = id_{A\uplus B}$, $*$ is the set-theoretic union. In $(\mathbf{FDVec}_K, \times)$, where $mix_{AB} = id_{A\times B}$, $*$ is the usual addition.

Note hat the operation $*$ is associative (owing to CR-*assoc* and CL-*assoc*) and commutative (owing to CR-*symm* and CL-*symm*). Now we turn towards proving that, in every Dummett category, $*$ is idempotent and therefore a semilattice.

Lemma 3.23. *Every Dummett category has a greatest morphism* $\bot \longrightarrow \top$, *and it is equal to* $\langle\rangle_\bot$ *and* $[]_\top$.

Proof. For every morphism $f : \bot \longrightarrow \top$, we have

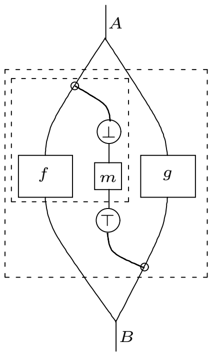
$$\begin{aligned} f &= id_\top \circ f \\ &= \langle\rangle_\top \circ f && \text{(by } \langle\rangle\textit{trivial}) \\ &\le \langle\rangle_\bot && \text{(by } \langle\rangle\textit{lax}). \end{aligned}$$

Dually, we get $f \le []_\top$. □

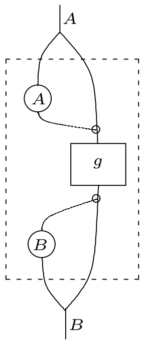
Lemma 3.24. *In every Dummett category, the laws below hold.*

$$f * g \le f \qquad\qquad f * g \le g$$

Proof. Without loss of generality, we show $f * g \le g$. We have

$f * g =$  ,

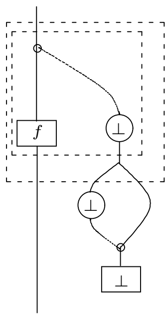
where the dashed boxes are only for visual guidance. By applying *Reduce*WL to the outermost dashed box, the above net is less or equal to

 .

By WR-*neutral* and WL-*neutral*, this is equal to g. □

Lemma 3.25. *For every morphism f of a Dummett category, it holds that $f * f = f$.*

Proof. The inequality $f * f \leq f$ follows directly from Lemma 3.24. For the converse, note that f is equal to



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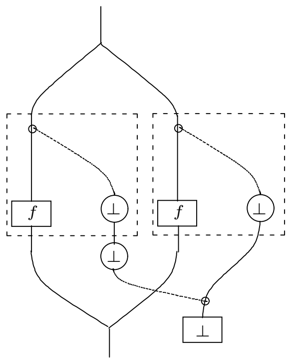
owing to WL-*neutral* and *Reduce*$\bot$. (The dashed boxes are only for visual guidance.) By applying *Reduce*CL to the subnet in the outermost dashed box, it follows that the above net is less or equal to



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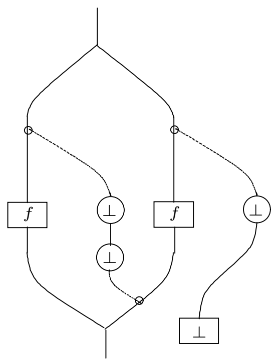
By empire rewiring, this is equal to



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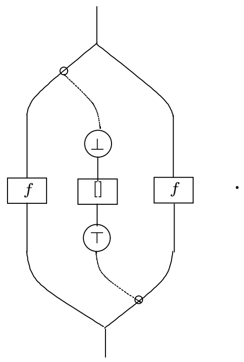
Now we apply *Reduce*$\bot$ to remove the "appendix", and expand the bottom thinning link in the sense of Definition 3.1; this yields



By Lemma 3.23, we have $[]_\top = m$, so the above net is equal to $f * f$. □

Remark 3.26. So the pre-Dummett category $(\mathbf{FDVec}_K, \times)$ cannot be a Dummett category, because the addition of vectors is not idempotent.

Lemma 3.27. *In very Dummett category, $*$ is monotonic in both arguments with respect to $\leq$.*

Proof. By definition of $*$ and the fact that $\oplus$, $\otimes$, and $\circ$ are monotonic with respect to $\leq$. □

Proposition 3.28. *In every Dummett category, the partial order $\leq$ agrees with the one induced by the semilattice structure—that is,*

$$f \leq g \iff f = f * g.$$

Proof. For the left-to-right implication, suppose that $f \leq g$. By Lemma 3.24, we have $f * g \leq f$; to see that $f \leq f * g$, consider

$$\begin{aligned} f &= f * f && \text{by Lemma 3.25} \\ &\leq f * g && \text{by Lemma 3.27.} \end{aligned}$$

The right-to-left implication holds because $f * g \leq g$ by Lemma 3.24. □

Composition does not generally preserve the semilattice structure—that is, Dummett categories are not generally semilattice-enriched. In fact, even classical categories are not generally semilattice-enriched. To see this, consider the classical category $(\mathbf{Rel}, \times)$. The operation $*$ is the set-theoretic intersection. We have

$$\begin{aligned} (x, z) \in h \circ (f * g) &\iff \exists y : (x, y) \in h \text{ and } (y, z) \in f \text{ and } (y, z) \in g \\ (x, z) \in (h \circ f) * (h \circ g) &\iff \exists y_1, y_2 : (x, y_1) \in h \text{ and } (y_1, z) \in f \\ &\qquad \text{and } (x, y_2) \in h \text{ and } (y_2, z) \in g. \end{aligned}$$

Obviously, the two relations differ for some f, g, and h. However, composition preserves $*$ in a lax way, and the same is true for $\otimes$ and $\oplus$:

Lemma 3.29. *In every Dummett category, the laws below hold.*

$$h \circ (f * g) \leq (h \circ f) * (h \circ g) \qquad (f * g) \circ k \leq (f \circ k) * (g \circ k) \qquad \circ\, lax$$
$$h \otimes (f * g) \leq (h \otimes f) * (h \otimes g) \qquad (f * g) \otimes k \leq (f \otimes k) * (g \otimes k) \qquad \otimes\, lax$$
$$h \oplus (f * g) \leq (h \oplus f) * (h \oplus g) \qquad (f * g) \oplus k \leq (f \oplus k) * (g \oplus k) \qquad \oplus\, lax$$

Proof. By Lemma 3.24, we have $f * g \leq f$ and $f * g \leq g$. Because $h \circ (-)$ is monotonic, we have $h \circ (f * g) \leq h \circ f$ and $h \circ (f * g) \leq h \circ g$. By Prop. 3.28, we get $h \circ (f * g) \leq (h \circ f) * (h \circ g)$. Similarly for the other five inequalities. □

Lemma 3.30 (Hasegawa)**.** *In a pre-Dummett category that satisfies the equation $id_B * id_B = id_B$ for every object B, the laws Reduce*WL *and Reduce*WR *are derivable.*

Proof. See the Appendix. □

The following theorem provides a characterization of Dummett categories in terms of unconditional inequalities (which can be stated as equalities owing to the semilattice structure):

Theorem 3.31. *To give a Dummett category is to give a pre-Dummett category satisfying the laws $\langle\rangle_\bot = []_\top$ and $f * f = f$, and, letting $\leq$ be the partial induced by the semilattice $*$, the laws Reduce*CL*, Reduce*CR*, $\circ$lax, $\otimes$lax, and $\oplus$lax.*

Proof. Every Dummett category satisfies the law $\langle\rangle_\bot = []_\top$ by Lemma 3.23 and the law $f * f = f$ by Lemma 3.25. By Prop. 3.28, the partial order of the Dummett category agrees with the order induced by the semilattice $*$. So *Reduce*CL and *Reduce*CR hold for the induced partial order, because they are required to hold in a Dummett category; the laws $\circ$*lax*, $\otimes$*lax*, and $\oplus$*lax* hold for the induced partial order, because they hold in a Dummett category, owing to Lemma 3.29.

Conversely, let **C** be a pre-Dummett category satisfying the equations $\langle\rangle_\bot = []_\top$. By Corollary 3.12, **C** is MIX with $m = \langle\rangle_\bot = []_\top$, and therefore $*$ is canonically defined. Now suppose that $f * f = f$, for every morphism f; let $\leq$ be the partial order induced by the semilattice $*$, and suppose that the laws *Reduce*CL, *Reduce*CR, $\circ$*lax*, $\otimes$*lax*, and $\oplus$*lax* hold. By Lemma 3.30, we have *Reduce*WL and *Reduce*WR. So, to see that we have a Dummett category, it remains to show that $\circ, \otimes, \oplus$ are monotonic in each argument. We shall show that $h \circ (-)$ is monotonic for every morphism h; the other cases are similar. So let $f \leq g$, that is, $f = f * g$. Using $\circ$*lax*, we get $h \circ f = h \circ (f * g) \leq (h \circ f) * (h \circ g) \leq h \circ g$. □

Theorem 3.32. *Let Σ be a signature containing AxWL, AxWR, AxCL, and AxCR. Let*

- *E be a set of equations on $Net(\Sigma)$,*
- *E' be the set of equations for pre-Dummett categories described in Tables 7 and 8 and their duals, and*
- *E'' be the set of equations (between nets) corresponding to the laws $\langle\rangle_\bot = []_\top$, $f * f = f$, Reduce*CL*, Reduce*CR*, $\circ$lax, $\otimes$lax, and $\oplus$lax, where $\leq$ is the partial order induced by the semilattice $*$.*

Then $Net_{E \cup E' \cup E''}(\Sigma)$ *is the free Dummett category generated by* Σ *and* E.
Similarly for classical categories.

Proof. This follows from Theorem 2.6, together with the fact that E' characterizes pre-Dummett categories and E'' characterizes Dummett categories, as stated in Theorem 3.31. □

3.4 Compact Dummett categories

We can think of a symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, \top)$ as a *compact* symmetric linearly distributive category, by which we mean that the two symmetric monoidal structures agree and distribution is the associativity $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$. If $\mathbf{C}$ has symmetric monoids and symmetric comonoids

$$\blacktriangle_A : A \longrightarrow A \otimes A \qquad \blacktriangledown_A : A \otimes A \longrightarrow A$$
$$\langle\rangle_A : A \longrightarrow \top \qquad []_A : \top \longrightarrow A,$$

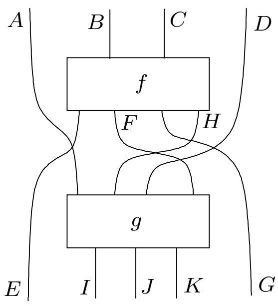
it forms a pre-Dummett category; in this section, we shall study the situation in which $\mathbf{C}$ is a Dummett category.

In particular, we shall present an axiomatization of such compact Dummett categories as compact pre-Dummett categories satisfying only one extra equality (Prop. 3.39). Moreover, we shall show how compact Dummett categories shed light on cut-reductions involving contraction (Prop. 3.38).

Some parts of this section are also required for our extended GoI construction, in § 4.

3.4.1 Nets for symmetric monoidal categories

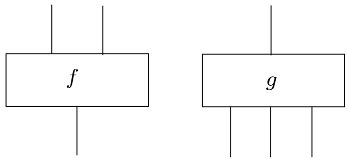
There are circuits that make sense in a symmetric monoidal category that do not make sense in every symmetric linearly distributive category. For example, the circuit



describes the morphism

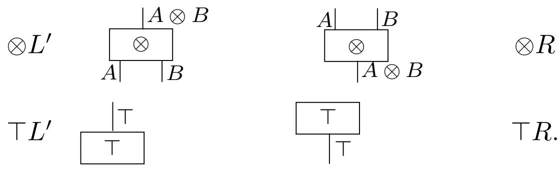
$$A \otimes B \otimes C \otimes D \xrightarrow{id \otimes f \otimes id} A \otimes E \otimes F \otimes G \otimes H \otimes D$$
$$\cong E \otimes A \otimes H \otimes D \otimes F \otimes G \xrightarrow{id \otimes g \otimes id} E \otimes I \otimes J \otimes F \otimes K \otimes G.$$

This does not make sense in every symmetric linearly distributive category, because f and g are connected by *two* wires (F and H), which requires having a morphism $F \oplus H \longrightarrow F \otimes H$. Also, juxtapositions like



make sense in a symmetric monoidal category (the semantics is $f \otimes g$), but not in every symmetric linearly distributive category. In fact, if f and g are circuits that describe morphisms in a symmetric monoidal category $\mathbf{C}$, and we connect any number of output ports of f with input ports of g (such that the types match), then the resulting circuit too describes a morphism in $\mathbf{C}$. We call this a *symmetric monoidal composition* of circuits. (However, we must not connect an output port with an input port of the *same* circuit, unless the category is traced. More about that in § 4.1.)

The links we shall use in nets for symmetric monoidal categories are



We already used $\otimes R$ and $\top R$ in nets for symmetric linearly distributive categories; not so the links $\otimes L'$ and $\top L'$. Both $\otimes L'$ and $\otimes R$ denote $id_{A \otimes B}$; they are useful for bundling multiple wires. Both $\top L'$ and $\top R$ denote $id_\top$; they are useful for removing wires of type $\top$.

When the category is compact closed, we also use the links $\neg\mathsf{L}$ and $\neg\mathsf{R}$.

Definition 3.33. The *symmetric monoidal nets* over a set of atomic types and a set $\mathcal{C}$ of components are the following circuits:

- Types are given by the grammar

  $$A, B ::= A \otimes B \mid \top \mid b,$$

  where b ranges over atomic types;

- Components are the links $\otimes L'$, $\otimes R$, $\top L'$, and $\top R$, and all elements of $\mathcal{C}$. In the compact closed case, we also have the links $\neg\mathsf{L}$ and $\neg\mathsf{R}$;

- If f and g are symmetric monoidal nets, then so is any symmetric monoidal composition of f and g.

There is an evident translation that sends nets for symmetric linearly distributive categories to symmetric monoidal nets. It translates formulæ by sending $\oplus$ to $\otimes$ and $\perp$ to $\top$. It translates circuits by replacing subcircuits according to the rules in Table 11. As is easy to see, this translation preserves semantics. That is, if f is a net for symmetric linearly distributive categories that denotes a morphism in a symmetric monoidal category, then the symmetric monoidal net that results from the translation denotes the same morphism.

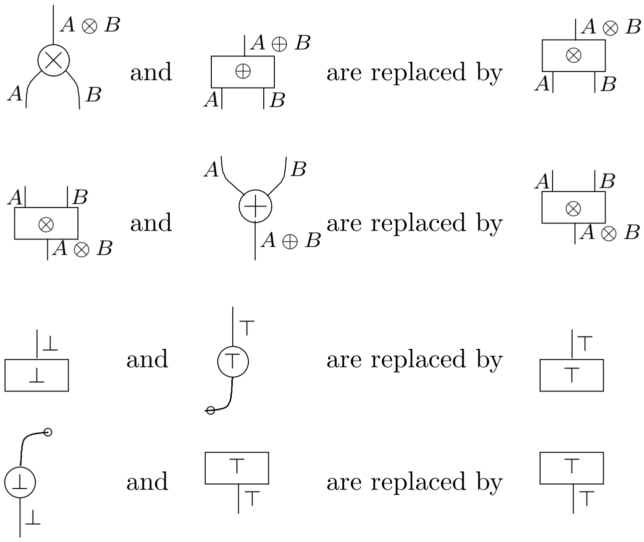
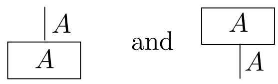
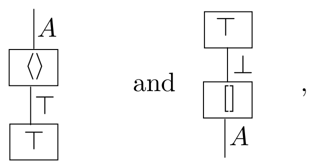


Table 11: Rules for translating nets for symmetric linearly distributive categories into nets for symmetric monoidal categories

When symmetric monoids and comonoids are present, we shall write



for

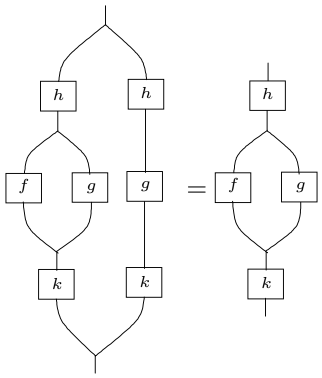


and we keep the notation for contractions given in Definition 3.1.

3.4.2 Characterizing compact Dummett categories by one equality

In this section, we show that a symmetric monoidal category with symmetric monoids an symmetric comonoids forms a Dummett category if and only if it satisfies the law below (Prop. 3.39).

$$(k \circ (f * g) \circ h) * (k \circ g \circ h) = k \circ (f * g) \circ h \tag{3}$$

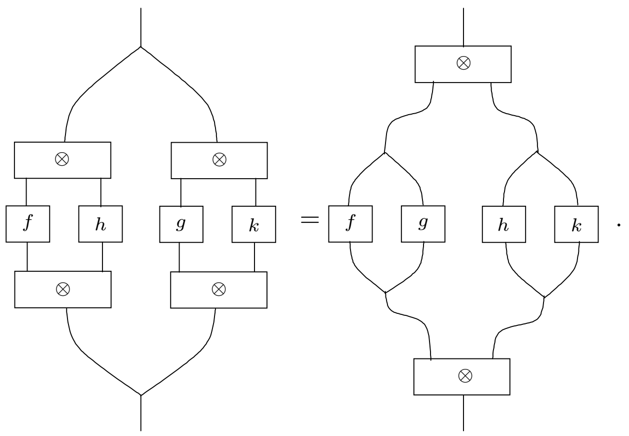


Moreover, we show how compact Dummett categories shed light on cut-reductions involving contraction (Prop. 3.38).

Lemma 3.34. *Every symmetric monoidal category with symmetric monoids and symmetric comonoids satisfies the laws below.*

$$f * e = f \qquad \text{where } e = [\,] \circ \langle\rangle \tag{4}$$
$$(f \otimes h) * (g \otimes k) = (f * g) \otimes (h * k) \tag{5}$$

The net-version of Equation 5 is



Proof. Equation 4 holds because

$$\begin{aligned} f * e &= \blacktriangledown \circ (f \otimes ([\,] \circ \langle\rangle)) \circ \blacktriangle \\ &= \blacktriangledown \circ (id \otimes [\,]) \circ (f \otimes id) \circ (id \otimes \langle\rangle) \circ \blacktriangle \\ &= f \qquad\qquad \text{(by } \langle\rangle \textit{neutral} \text{ and } [\,]\textit{neutral}\text{).} \end{aligned}$$

Equation 5 it holds owing to CL*pointwise* and CR*pointwise*. □

Lemma 3.35. *In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, the law $g * g = g$ holds.*

Proof. Let $k = id$, $h = id$, and $f = e$ in Equation 3. □

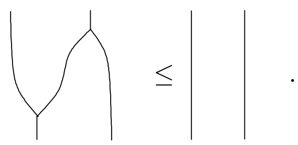
Lemma 3.36. *In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3,* $\circ$ *and* $\otimes$ *are monotonic in both arguments with respect to the partial order induced by the hom-semilattice* $*$.

Proof. The monotonicity of $\circ$ follows from Equation 3: for $f \leq g$ (i.e., $f*g = id$) $k\circ f\circ h \leq k\circ g\circ h$ holds because $(k\circ f\circ h)*(k\circ g\circ h) = (k\circ(f*g)\circ h)*(k\circ g\circ h) = k\circ(f*g)\circ h = k\circ f\circ h$. To see the monotonicity of $\otimes$, suppose that $f \leq g$. Then $f\otimes h \leq g\otimes h$ holds because

$$\begin{aligned}(f\otimes h)*(g\otimes h) &= (f*g)\otimes(h*h) && \text{(by Equation 5 of Lemma 3.34)}\\ &= (f*g)\otimes h = f\otimes g.\end{aligned}$$

□

Lemma 3.37. *In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, it holds that* $(\blacktriangledown\otimes id)\circ(id\otimes\blacktriangle)\leq id$.



Proof. We have

$(\blacktriangledown\otimes id)\circ(id\otimes\blacktriangle)$

$= (\blacktriangledown\otimes id)\circ(id\otimes id\otimes id)\circ(id\otimes\blacktriangle)$

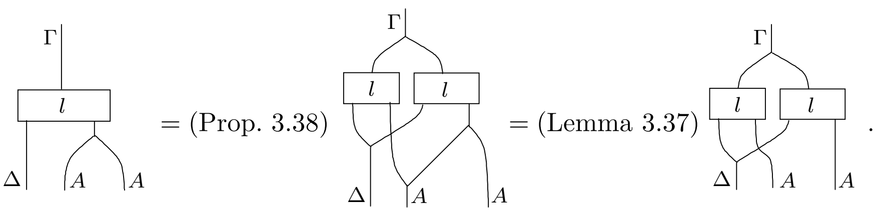
$\leq (\blacktriangledown\otimes id)\circ(id\otimes e\otimes id)\circ(id\otimes\blacktriangle)$ (because, by previous lemmas, e is the neutral/greatest element and $\circ$ and $\otimes$ are monotonic in both arguments).

$= (\blacktriangledown\otimes id)\circ(id\otimes([]\circ\langle\rangle)\otimes id)\circ(id\otimes\blacktriangle)$

$= id$ (by $[]$*neutral* and $\langle\rangle$*neutral*)

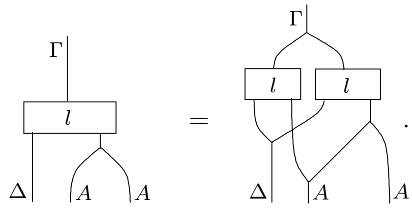
□

Proposition 3.38 below implies that, in every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, the law *Reduce*CL can be split into two steps. The first step is the *equality* stated in the proposition, and the second step is an *inequality* which results from applying Lemma 3.37:



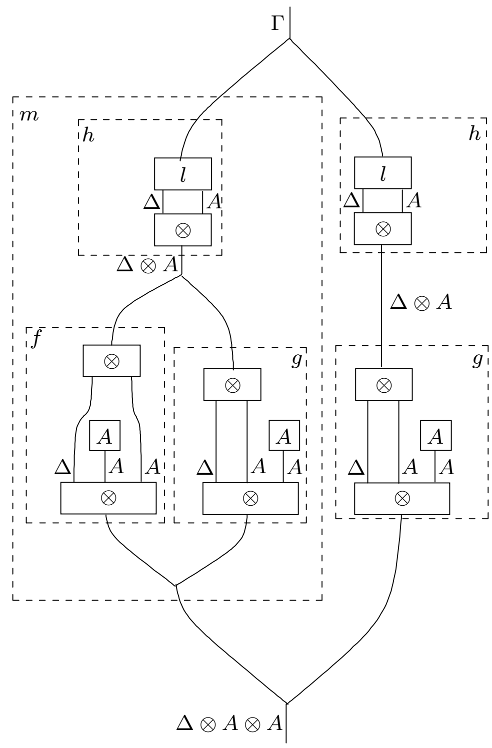
(However, the net in the middle is not generally the denotation of a sequent proof.)

Proposition 3.38. *In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, it holds that*



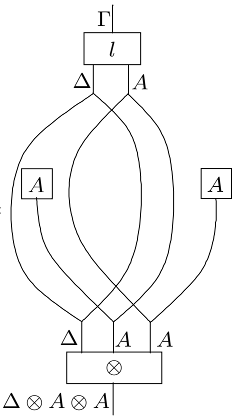
Proof. We use Equation 3 with $k = id$ and f, g, h such that the left-hand side looks as follows:

$n \quad =$

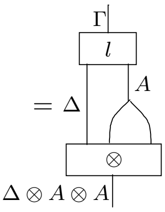


. (6)

First, we focus on the morphism given by the subnet m, which is the right-hand side of Equation 3. We have

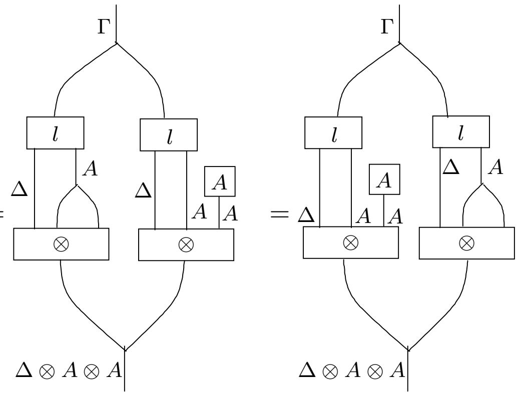
$m =$ 

(by CL-*pointwise*, CR-*pointwise*, and *Reduce*$\otimes$)

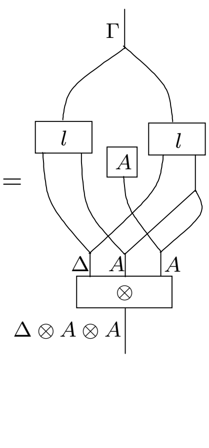
$=$ 

(by WR-*neutral* and because $\blacktriangledown \circ \blacktriangle = id * id = id$ by Lemma 3.35).

By simplifying m accordingly in the net n, we get

$n =$ 

(by CL-*symm* and CR-*symm*)

$=$ 

(by CR-*pointwise*).

By WR-*neutral*, this is equal to the right-hand side of Equation 6. So the right-hand side of Equation 6 is equal to m, which by Equation 3 is equal to n, which as just shown is equal to the left-hand side of Equation 6. □

Proposition 3.39 (Hasegawa)**.** *A symmetric monoidal category with symmetric monoids and symmetric comonoids forms a Dummett category if and only if Equation 3 holds.*

Proof. Let $\mathbf{C}$ be a symmetric monoidal category with monoids and comonoids. For the left-to-right direction, suppose that $\mathbf{C}$ forms a Dummett category. The $\leq$ direction of Equation 3 is trivial, because the semilattice operation $*$ is the greatest lower bound with respect to $\leq$. The $\geq$ direction holds because

$$\begin{aligned} k \circ (f * g) \circ h &= k \circ ((f * g) * g) \circ h \\ &= k \circ \blacktriangledown \circ ((f * g) \otimes g) \circ \blacktriangle \circ h \\ &\leq \blacktriangledown \circ (k \otimes k) \circ ((f * g) \otimes g) \circ (h \otimes h) \circ \blacktriangle \quad \text{(by } \blacktriangle \mathit{lax} \text{ and } \blacktriangledown \mathit{lax}) \\ &= (k \circ (f * g) \circ h) * (k \circ g \circ h). \end{aligned}$$

For the right-to-left direction of the proposition, suppose that $\mathbf{C}$ satisfies Equation 3. The monotonicity of $\circ$ and $\otimes$ in both arguments follows from Lemma 3.36. The law *Reduce*CL holds because of Prop. 3.38 and Lemma 3.37. Dually for *Reduce*CR. □

3.4.3 Dummett categories with finite biproducts

In this section, we discuss the special case of compact Dummett categories where the tensor/cotensor is a biproduct. That is, Dummett categories in which $\otimes = \oplus$ is

- the cartesian product, with diagonals and projections given by $\blacktriangle$ and $\langle\rangle$,
- the cartesian coproduct, with codiagonals and coprojections given by $\blacktriangledown$ and $[]$, and

$\bot = \top$ is the zero (i.e., initial and terminal) object.

Such categories have a very simple axiomatization:

Proposition 3.40 (Hasegawa)**.** *A category with finite biproducts forms a Dummett category (with the biproduct as the tensor/cotensor) if and only if the equation $\blacktriangle \circ \blacktriangledown = id$ holds.*

Proof. Let $\mathbf{C}$ be a category with finite biproducts. If $\mathbf{C}$ is a Dummett category, we have

$$\begin{aligned} \blacktriangledown \circ \blacktriangle &= id * id \\ &= id \qquad \text{(by Lemma 3.25).} \end{aligned}$$

If **C** satisfies the equation $\blacktriangledown \circ \blacktriangle$, then Equation 3 holds because

$$\begin{aligned}
&(k \circ (f * g) \circ h) * (k \circ g \circ h)\\
&= \blacktriangle \circ ((k \circ (\blacktriangledown \circ (f \times g) \circ \blacktriangle) \circ h) \times (k \circ g \circ h)) \circ \blacktriangledown\\
&= k \circ \blacktriangle \circ (f \times (\blacktriangledown \circ \blacktriangle)) \circ (id \times g) \circ \blacktriangledown \circ h && \text{(by calculations that hold in every category with biproducts)}\\
&= k \circ \blacktriangle \circ (f \times id) \circ (id \times g) \circ \blacktriangledown \circ h\\
&= k \circ (f * g) \circ h.
\end{aligned}$$

□

Remark 3.41. We have already seen in Example 3.16 that $(\mathbf{Rel}, \uplus)$ is a Dummett category with finite biproducts. Prop. 3.40 makes clear that we could have checked this simply by verifying the equation $\blacktriangledown \circ \blacktriangle = id$, which obviously holds.

When dealing with categories with finite biproducts, we follow common practice and write

- $\oplus$ (not $\otimes$) for the biproduct,
- $\perp$ (not $\top$) for the zero object,
- $+$ instead of $*$, and
- 0 instead of $e = [] \circ \langle\rangle$.

Given objects $A_1, \cdots, A_n$ and $B_1, \cdots, B_m$, and maps $f_{lk} : A_l \longrightarrow B_k$ for $l \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, we write

$$\begin{pmatrix} f_{11} & \cdots & f_{n1} \\ \vdots & & \vdots \\ f_{1m} & \cdots & f_{nm} \end{pmatrix}$$

for the unique morphism $f : A_1 \oplus \cdots \oplus A_n \longrightarrow B_1 \oplus \cdots \oplus B_m$ such that $\pi_k \circ f \circ \iota_l = f_{lk}$. As can easily be seen, composition agrees with matrix multiplication.

The homomorphism analysis carried out for $(\mathbf{Rel}, \uplus)$ in § 3.2.1 can be generalized to *all* Dummett categories with finite biproducts. Let $f : \Gamma \longrightarrow \Delta \oplus A$ be a homomorphism of a category with finite biproducts. Both the redex and the reduct of *Reduce*CL turn out to be

$$\left(\begin{array}{c|c} & \Gamma \\ \hline \Delta & f_{\Gamma\Delta} \\ A & f_{\Gamma A} \\ A & f_{\Gamma A} \end{array}\right).$$

So every $f : \Gamma \longrightarrow \Delta \oplus A$ is a parametrized cosemigroup homomorphism. The redex and reduct of *Reduce*WL turn out to be given by the matrices

$$\left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & f_{\Gamma\Delta} & 0 \\ A & 0 & id_B \end{array}\right) \text{ and } \left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & 0 & 0 \\ A & 0 & id_B \end{array}\right),$$

respectively. So f is a parametrized copointed homomorphism if and only if $f_{\Gamma\Delta} = 0$.

4 Geometry of interaction in the presence of weakening and contraction

The *Geometry of Interaction* (GoI) was introduced by Girard (Girard 1989, Girard 1990, Girard 1995) in the late 1980s in the context of modelling the dynamics of cut elimination in (classical) linear logic (Girard 1987). The aim was to capture the essential structure of the proof theory of cut elimination while avoiding the semantically inessential aspects of the syntax.

A categorical approach to GoI, based on domain theory and arising from the construction of a categorical model of linear logic, has been described in (Abramsky & Jagadeesan 1994). Some years later, Abramsky *et al.* presented what can be seen as a general form of the Geometry of Interaction: a compact closed category is constructed from a traced symmetric monoidal category (Abramsky 1996, Abramsky, Haghverdi & Scott 2001). This construction also appeared in (Joyal, Street & Verity 1996).

Many of the ideas contributing to these developments have also been described by Hyland. Beginning in lectures dating from 1992, Hyland has described a range of ideas, from the construction of compact closed categories from what are now called traced monoidal categories, and explaining GoI as a matter of interpreting derivations with cuts in such categories, through to a recent invited paper (Hyland 2004) in which interpretations of contraction and weakening in traced categories with biproducts are also considered.

Recently, Haghverdi and Scott have considered a GoI semantics for multiplicative exponential linear logic based on "unique decomposition categories" (Haghverdi & Scott to appear). Their objective is distinct from ours in that they are not concerned with classical logic and that they give an abstract account of Girard's original "untyped" notion of GoI.

The main contribution of this section is an extended GoI construction that sends a traced Dummett category to a classical category. This shows that GoI works in the presence of weakening and contraction, even with respect to the partial order that models cut-reductions.

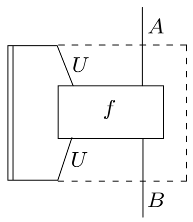
In § 4.1, we introduce traced symmetric monoidal categories as symmetric MIX categories all of whose objects are traced. In § 4.2, we review the traditional construction of a compact closed category from a traced symmetric monoidal category and present it in terms of nets. In § 4.3, we extend that construction to traced Dummett categories. In § 4.4, we study the special case where the starting point of the extended GoI construction is a traced Dummett category with finite biproducts, and carry out a homomorphism analysis in the sense of § 3.2.1.

4.1 Traced symmetric MIX categories

In this section, we recall the notion of *traced object* in a symmetric MIX category from (Blute et al. 2000), because our extended GoI construction starts with a Dummett (and therefore symmetric MIX) category $\mathbf{C}$ all of whose objects are traced (in a compatible way). But, as we shall see, a symmetric MIX category in which every object is traced is compact in the sense that all maps $mix_{AB} : A \otimes B \longrightarrow A \oplus B$ and $m : \bot \longrightarrow \top$ are isomorphisms. For the sake of presentation, we shall assume that these isomorphisms are identities, so

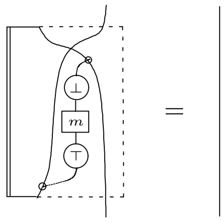
C is simply a traced monoidal category with the extra structure required for a Dummett category. We take this detour via traced objects, as opposed to introducing traced symmetric monoidal categories straight away, to show that assuming compactness in the presence of a trace implies no loss of generality.

An object U of a symmetric MIX category **C** is said to have a *trace* if there is a family of functions $tr_U^{AB} : \mathbf{C}(U \otimes A, U \otimes B) \longrightarrow \mathbf{C}(A, B)$ satisfying certain equations that we shall present shortly. Following (Blute et al. 2000), we write

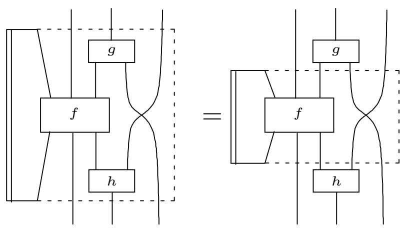


for the net that represents $tr(f : U \otimes A \longrightarrow U \oplus B)$. We think of the trace of "feedback along U". The dashed box indicates the scope of the trace.

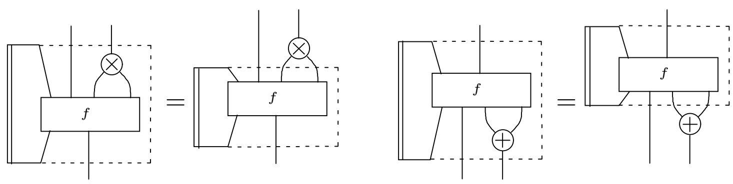
The equational laws, presented in terms of nets, are



Yanking



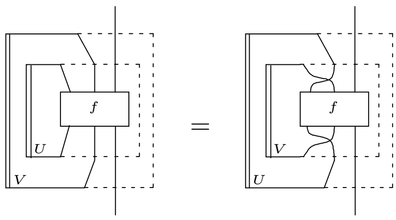
Tightening



Superposing

The left net in the Yanking law describes a trace over the twisted version of $m : U \otimes U \longrightarrow U \oplus U$, that is, over the map $mix \circ \sigma_\otimes = \sigma_\oplus \circ mix : U \otimes U \longrightarrow U \oplus U$. The Tightening law lives up to its name and describes how the scope of the trace can be tightened. The Superposing law (called "Superposing (ii)" in (Blute et al. 2000)) explains how the scope of the trace can be tightened when the links $\otimes L$ and $\oplus R$ are involved. The categorical-style versions of these laws and a more detailed discussion can be found in (Blute et al. 2000).

Now let U and V be objects of a symmetric MIX category $\mathbf{C}$, with trace operators tr_U and tr_V, respectively. These traces are called *compatible* if the equation



Compatibility

holds for every $f : U \otimes V \otimes A \longrightarrow U \oplus V \oplus B$. Note that compatibility, like Tightening and Superposing, is about manipulating the scopes of traces.

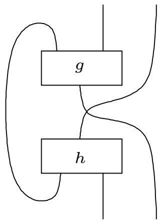
Now let $\mathbf{C}$ be a symmetric MIX category *some* of whose objects have a trace, and suppose all those traces are compatible. Then it is not hard to show that the laws for Tightening, Superposing, and compatibility together imply that the scope of a trace can be extended and contracted arbitrarily (as long as the net stays syntactically correct), so the dashed boxes become unnecessary.

By Prop. 10 of (Blute et al. 2000), every traced object U of a symmetric MIX category $\mathbf{C}$ is in the *core* of $\mathbf{C}$—that is, for every object A, the map $mix : U \otimes A \longrightarrow U \oplus A$ has an inverse (given by the trace over the distribution $\delta : U \otimes (U \oplus A) \longrightarrow U \oplus (U \otimes A)$). By Prop. 11 of (Blute et al. 2000), if either $\bot$ or $\top$ has a trace then $m : \bot \longrightarrow \top$ has an inverse (given by the trace over the map $U \otimes \top \cong U \cong U \oplus \bot$, where $U = \bot$ or $U = \top$). So, if every object of $\mathbf{C}$ has a trace, as will be the case in our extended GoI construction, then all maps $mix_{AB} : A \otimes B \longrightarrow A \oplus B$ and $m : \bot \longrightarrow \top$ have inverses. For the sake of presentation, we shall assume that mix and m are identities. Thus, we recover the original notion of traced monoidal category as a symmetric monoidal category in which every object is traced such that any two traces are compatible. (For a more detailed discussion of this fact, see (Blute et al. 2000).)

So we shall use the symmetric-monoidal nets described in § 3.4.1.

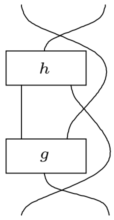
We shall further simplify the notation for the trace by replacing

The right-hand circuit introduces no unwanted ambiguity: if the loop is part of a cycle (i.e., there is a connection between the loop's entry point into f and the loop's exit point from f), the loop necessarily stands for a trace; if not, then the loop's entry point is connected with some subnet g of f, and the exit point is connected with some subnet h of f, such that g and h are not connected:



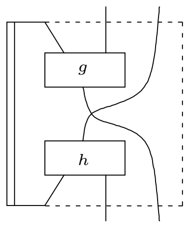
.

This can be rewritten as

$k =$ 

.

The scenario where the loop is a trace, presented in our old notation, is



.

By Tightening and Yanking, this is equivalent to k.

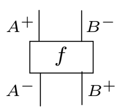
So, in the presence of a trace, we can allow wires from an output port to an input port of the *same* circuit. So *every* circuit built from $\otimes L'$, $\otimes R$, $\top L'$, and $\top R$ and other components forms a *traced symmetric monoidal net.* The equations for the trace, even yanking, are built into the syntax of the nets. As in the untraced case, the only equations needed are *Reduce*$\otimes$, *Reduce*$\top$, *Expand*$\otimes$, and *Expand*$\top$.

4.2 The "traditional" GoI construction

In this section, we describe the traditional construction of a compact closed category from a traced symmetric monoidal category. Our point here is to express this well-known construction in terms of the traced symmetric monoidal nets described in § 4.1; this helps calculations in § 4.3.

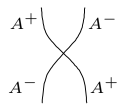
Given a traced symmetric monoidal category $\mathbf{C}$, the category $\mathcal{G}(\mathbf{C})$ is defined below.

- Objects are pairs (A^+, A^-) of objects of $\mathbf{C}$.
- A morphism $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ of $\mathcal{G}(\mathbf{C})$ is a morphism $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ of $\mathbf{C}$:

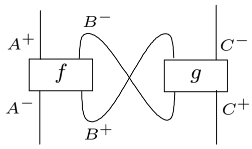


.

- The identity on (A^+, A^-) is the twist map $A^+ \otimes A^- \cong A^- \otimes A^+$ of $\mathbf{C}$:

$id_{(A^+,A^-)} =$  .

- The composition of morphisms $(A^+, A^-) \xrightarrow{f} (B^+, B^-) \xrightarrow{g} (C^+, C^-)$ is given by the net

$g \circ f =$  .

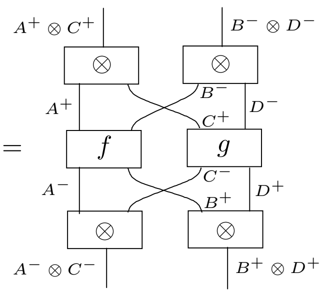
Proposition 4.1. $\mathcal{G}(\mathbf{C})$ *is a compact closed category.*

The proof of this theorem is well-known; however, we shall present our own proof to familiarize the reader with our usage of nets.

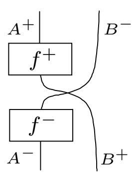
Proof. For objects (A^+, A^-) and (B^+, B^-), we define

$$(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-).$$

The tensor unit of $\mathcal{G}(\mathbf{C})$ is $(\top, \top)$. For morphisms $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (C^+, C^-) \longrightarrow (D^+, D^-)$, we define

$f \otimes g =$  .

That $\otimes : \mathcal{G}(\mathsf{C}) \times \mathcal{G}(\mathsf{C}) \longrightarrow \mathcal{G}(\mathsf{C})$ is a functor follows immediately from the reduction and expansion rules for nets. Now for an auxiliary definition: for morphisms $f^+ : A^+ \longrightarrow B^+$ and $f^- : A^- \longrightarrow B^-$ of $\mathbf{C}$, we define

$f^+ \times f^- =$  .

Note that $\times$ forms a (faithful) functor $\mathbf{C} \times \mathbf{C} \longrightarrow \mathcal{G}(\mathbf{C})$ that preserves the monoidal product, that is

$$(f^+ \otimes g^+) \times (f^- \otimes g^-) = (f^+ \times f^-) \otimes (g^+ \times g^-).$$

The symmetric-monoidal isomorphisms of $\mathcal{G}(\mathbf{C})$ are $\alpha \times \alpha^{-1}$, $\lambda \times \lambda^{-1}$, $\rho \times \rho^{-1}$, and $\sigma \times \sigma^{-1}$. Showing their naturality is straightforward. Their coherence follows immediately from the coherence of the corresponding maps of $\mathbf{C}$ and the fact that $\times$ is a functor that preserves $\otimes$.

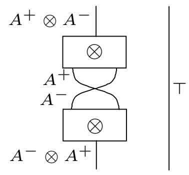
We define

$$(A^+, A^-)^\perp = (A^-, A^+);$$

the map

$$\gamma^R : (A^+, A^-) \otimes (A^+, A^-)^\perp = (A^+ \otimes A^-, A^- \otimes A^+) \longrightarrow (\top, \top) = \top$$

is given by



;

dually for τ^R, and symmetrically for γ^L and τ^L. Checking the two equations required for γ and τ is a laborious routine verification. □

4.3 The GoI construction extended to traced Dummett categories

Our extended GoI construction starts with a Dummett category in which *every* object has a trace, such that the traces on any two objects are compatible. As explained in § 4.1, this causes $m : \bot \longrightarrow \top$ and $mix_{AB} : A \otimes B \longrightarrow A \oplus B$ to be isomorphisms, and to make our lives a little easier, we assume that they are identities. So we define:

Definition 4.2. A *traced Dummett category* is a traced symmetric monoidal category together with symmetric comonoids and symmetric monoids that satisfies the conditions of a Dummett category.

Example 4.3. $(\mathbf{Rel}, \uplus)$ is a traced Dummett category: the trace of a relation

$$\left(\begin{array}{c|cc} & U & A \\ \hline U & f_{UU} & f_{AU} \\ B & f_{UB} & f_{AB} \end{array}\right) : U \otimes A \longrightarrow U \otimes B$$

is $f_{AB} \cup f_{BU} \circ f^*_{UU} \circ f_{UA} : A \longrightarrow B$, where f^*_{UU} is the reflexive-transitive closure of f_{UU}.

Theorem 4.4. *If* $\mathbf{C}$ *be a traced Dummett category, the compact closed category* $\mathcal{G}(\mathbf{C})$ *is a classical category.*

Proof. The multiplication

$$(A^+ \otimes A^+, A^- \otimes A^-) = (A^+, A^-) \otimes (A^+, A^-) \longrightarrow (A^+, A^-)$$

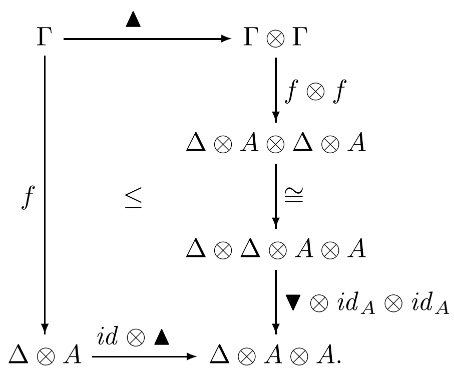
is $\blacktriangledown_{A^+} \times \blacktriangle_{A^-}$, and the unit

$$(\top, \top) \longrightarrow (A^+, A^-)$$

is $[]_{A^+} \times \langle\rangle_{A^-}$. The laws $\blacktriangledown$*assoc*, $[]$*neutral*, $\blacktriangledown$*symm*, $\blacktriangledown$*pointwise*, $[]$*pointwise*, and $[]$*trivial*, result from the corresponding laws for the monoids *and* comonoids of $\mathbf{C}$ and the fact that the functor $\times : \mathbf{C} \times \mathbf{C} \longrightarrow \mathcal{G}(\mathbf{C})$ preserves $\otimes$. Dually, we obtain the laws for comonoids on $\mathcal{G}(\mathbf{C})$. So $\mathcal{G}(\mathbf{C})$ is a pre-Dummett category with negation. To turn it into a classical category, we define $f \leq g : (A^+, A^-) \longrightarrow (B^+, B^-)$ if and only if $f \leq g : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ holds in $\mathbf{C}$. The monotonicity of $\leq$ with respect to $\otimes$ and $\circ$ in $\mathcal{G}(\mathbf{C})$ follows from the same kind of monotonicity of $\leq$ in $\mathbf{C}$.

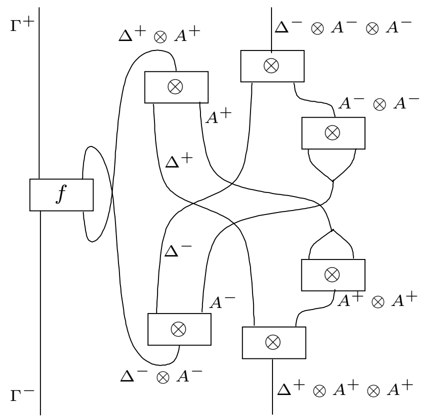
As can be easily checked, we have $\blacktriangledown \circ \blacktriangle = id$ in $\mathcal{G}(\mathbf{C})$, that is, $id * id = id$; so by Lemma 3.30, we have *Reduce*WL and *Reduce*WR.

It remains to check *Reduce*CL and *Reduce*CR. We check *Reduce*CL. In a compact pre-Dummett category, *Reduce*CL is

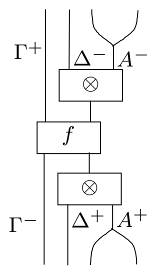


(7)

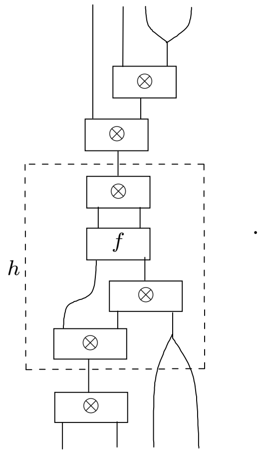
As follows directly from the definition of $\mathcal{G}(\mathbf{C})$, the left-bottom leg is



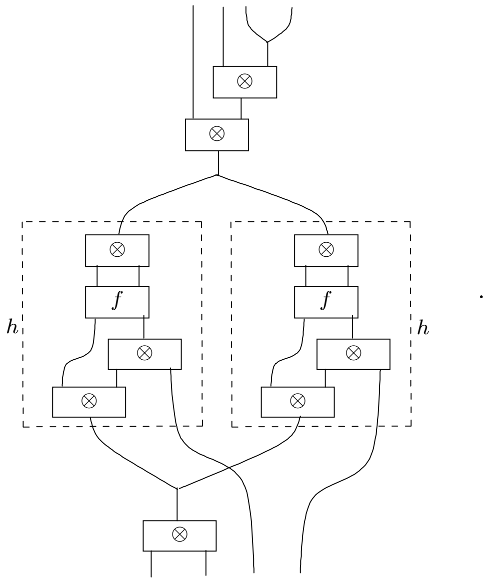
Optimizing the layout and removing inessential outermost $\otimes$-links yields



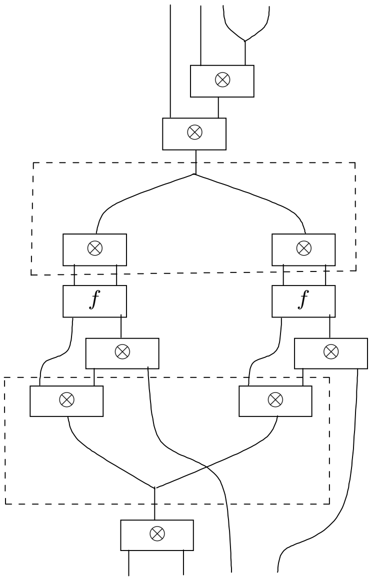
Now we apply two cut-reductions for $\otimes$ backwards and focus on the subnet h:



Applying the law *Reduce*CL to h yields

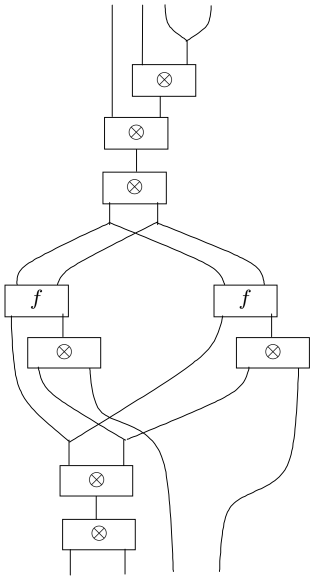


Now we forget h and focus on two new subnets:



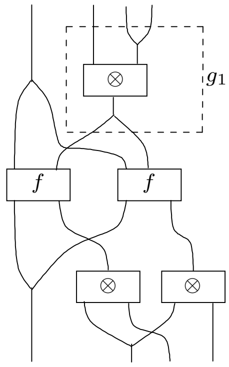
.

Applying the law CL-*pointwise* to the upper subnet and CR-*pointwise* to the lower subnet yields



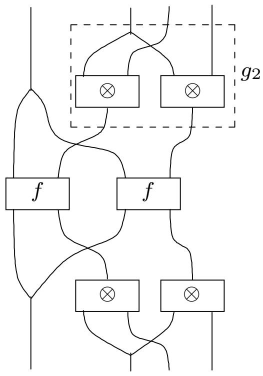
.

After eliminating the two logical cuts, we get



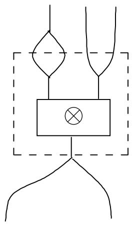
.

(The subnet g_1 is only for later reference.) As follows directly from the definition of $\mathcal{G}(\mathbf{C})$, the top-right leg of Diagram 7 is



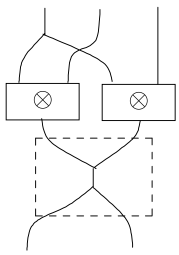
.

The last two nets differ only in the subnets g_1 and g_2, so it remains to show that they are equivalent. Because we have $id = id * id = \blacktriangledown \circ \blacktriangle$ in $\mathbf{C}$ (Lemma 3.25), g_1 is equivalent to



.

Applying the law CR-*pointwise* to the marked subnet yields

$k =$  .

The subnet in the dashed box is $\blacktriangle \circ \blacktriangledown$. By Lemma A.2 (applied to the compact case where $m_{AB} = id_{AB}$), we have $\blacktriangle \circ \blacktriangledown \leq id$. So $k \leq g_2$. $\square$

4.4 GoI for traced categories with finite biproducts

In this section, we study our extended GoI construction in the case where the traced Dummett category $\mathbf{C}$ is a category with finite biproducts. (Recall that, by Prop. 3.40, a category with finite biproducts is a Dummett category if and only if the equation $\blacktriangledown \circ \blacktriangle = id$ holds.) Using the matrix presentation of morphisms, which is available in the presence of biproducts (recall § 3.4.3), we obtain a precise characterization of parametrized (co)pointed homomorphisms and parametrized (co)semigroup homomorphisms (Prop. 4.5). Thus, we gain a complete understanding of the denotational change caused by *Reduce*CL/*Reduce*CR and *Reduce*WL/*Reduce*WR in $\mathcal{G}(\mathbf{C})$. We shall also see that (unparametrized) monoid homomorphism and comonoid homomorphisms are the same in $\mathcal{G}(\mathbf{C})$ (Corollary 4.6), and all denotations of positive (i.e., negation-free) derivations or nets are monoid/comonoid homomorphisms (Corollary 4.7).

Without loss of generality, we shall focus on *Reduce*WL and *Reduce*CL. Let $f : \Gamma \longrightarrow \Delta \otimes A$ be a morphism of $\mathcal{G}(\mathbf{C})$, where $\Gamma = (\Gamma^+, \Gamma^-)$, $\Delta = (\Delta^+, \Delta^-)$, and $A = (A^+, A^-)$. We want to characterize when f is a parametrized copointed homomorphism (resp. parametrized cosemigroup homomorphism), that is, when the laws *Reduce*WL (resp. *Reduce*CL) hold as equalities. In $\mathbf{C}$, we have $f : \Gamma^+ \otimes \Delta^- \otimes A^- \longrightarrow \Gamma^- \otimes \Delta^+ \otimes A^+$; owing to the biproducts, f can be presented as a 3×3-matrix

$$f = \left(\begin{array}{c|ccc} & \Gamma^+ & \Delta^- & A^- \\ \hline \Gamma^- & f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} \\ \Delta^+ & f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} \\ A^+ & f_{\Gamma A} & f_{\Delta A} & f_{AA} \end{array}\right).$$

Proposition 4.5. *Let $\mathbf{C}$ be a traced category with finite biproducts satisfying the law $\blacktriangledown \circ \blacktriangle = id$. Let*

$$f : (\Gamma^+, \Gamma^-) \longrightarrow (\Delta^+, \Delta^-) \otimes (A^+, A^-)$$

be a morphism of $\mathcal{G}(\mathbf{C})$. Then f is

- *a parametrized copointed homomorphism if and only if it has the form*

$$f = \left(\begin{array}{c|ccc} & \Gamma^+ & \Delta^- & A^- \\ \hline \Gamma^- & 0 & 0 & f_{A\Gamma} \\ \Delta^+ & 0 & 0 & f_{A\Delta} \\ A^+ & f_{\Gamma A} & f_{\Delta A} & f_{AA} \end{array}\right);$$

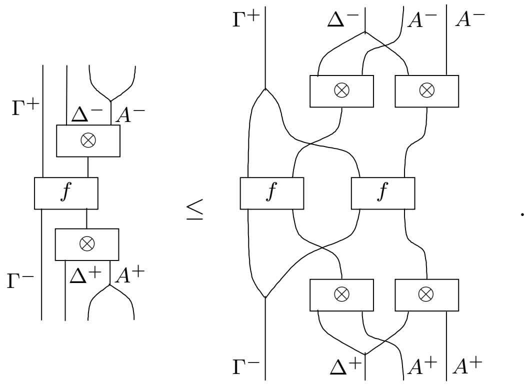
- *a parametrized cosemigroup homomorphism if and only if it has the form*

$$f = \left(\begin{array}{c|ccc} & \Gamma^+ & \Delta^- & A^- \\ \hline \Gamma^- & f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} \\ \Delta^+ & f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} \\ A^+ & f_{\Gamma A} & f_{\Delta A} & 0 \end{array}\right).$$

Dually for pointed homomorphisms and semigroup homomorphisms.

Proof. By definition, f is a parametrized cosemigroup homomorphism if it satisfies *Reduce*CL as an equality. As observed in the proof of Theorem 4.4, in

$\mathcal{G}(\mathbf{C})$, the law *Reduce*CL boils down to

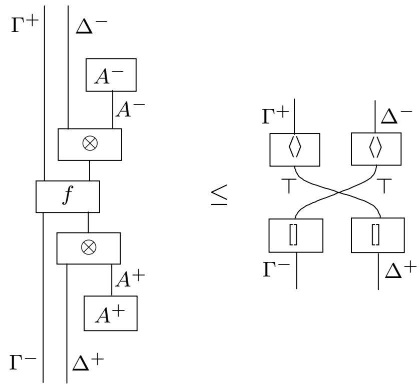


Translating this into matrix form yields

$$\begin{pmatrix} f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} & f_{A\Gamma} \\ f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} & f_{A\Delta} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} & f_{AA} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} & f_{AA} \end{pmatrix} \leq \begin{pmatrix} f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} & f_{A\Gamma} \\ f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} & f_{A\Delta} \\ f_{\Gamma A} & f_{\Delta A} & 0 & f_{AA} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} & 0 \end{pmatrix}$$

This is an equality if and only if $f_{AA} = 0$.

By definition, f is a parametrized copointed homomorphism if it satisfies *Reduce*WL as an equality. As it turns out, *Reduce*WL boils down to



in $\mathbf{C}$. Translating this into matrix form yields

$$\left(\begin{array}{c|cc} & \Gamma^+ & \Delta^- \\ \hline \Gamma^- & f_{\Gamma\Gamma} & f_{\Delta\Gamma} \\ \Delta^+ & f_{\Gamma\Delta} & f_{\Delta\Delta} \end{array}\right) \leq \left(\begin{array}{c|cc} & \Gamma^+ & \Delta^- \\ \hline \Gamma^- & 0 & 0 \\ \Delta^+ & 0 & 0 \end{array}\right).$$

This is an equality if and only if $f_{\Gamma\Gamma}$, $f_{\Delta\Gamma}$, $f_{\Gamma\Delta}$, and $f_{\Delta\Delta}$ are zero. ☐

Corollary 4.6. *Let $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ be a morphism in a traced category with finite biproducts satisfying the equation $\blacktriangledown \circ \blacktriangle = id$. Let*

$$f = \left(\begin{array}{c|cc} & A^+ & B^- \\ \hline A^- & f_{AA} & f_{BA} \\ B^+ & f_{BA} & f_{BB} \end{array}\right) : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$$

be the matrix presentation of f in $\mathbf{C}$. Then the following are equivalent:

- $f_{AA} = 0$;
- f *is a copointed homomorphism;*
- f *is a semigroup homomorphism.*

Dually, the following are equivalent:

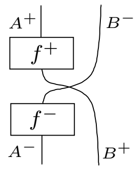
- $f_{BB} = 0$;
- f *is a pointed homomorphism;*
- f *is a cosemigroup homomorphism.*

In particular, f *is a monoid homomorphisms if and only if it is a comonoid homomorphism, which is the case if*

$$f = \left(\begin{array}{c|cc} & A^+ & B^- \\ \hline A^- & 0 & f_{BA} \\ B^+ & f_{BA} & 0 \end{array} \right),$$

that is, if f *is of the form*

$$f^+ \times f^- = $$



Corollary 4.7. *Let* $\mathbf{C}$ *be a traced category with finite biproducts that satisfies the equation* $\blacktriangledown \circ \blacktriangle = id$*. Then all denotations of positive (i.e., negation-free) derivations or nets in* $\mathcal{G}(\mathbf{C})$ *are monoid/comonoid homomorphisms.*

Proof. The denotations of all axioms except Ax¬L and Ax¬R are of the form $f^+ \times f^-$, and denotations of the form $f^+ \times f^-$ are closed under $\wedge$L, $\vee$R, $\top$L, $\bot$R, EL, ER, and Cut. (Note also that the denotations of Ax¬L (resp. Ax¬R) are γ^L (resp. τ^R), as defined in the proof of Prop. 4.1, and it is clear that they are not the form $f^+ \times f^-$.) □

5 Directions for future work

More non-compact classical categories. We have presented classical categories with non-trivial hom-sets (i.e., hom-sets with more than one element)—for example, $(\mathbf{Rel}, \times)$ and $\mathcal{G}(\mathbf{C})$, where $\mathbf{C}$ is a Dummett category (e.g. $(\mathbf{Rel}, \uplus)$). However, these models are compact—that is, $\otimes = \oplus$. On the other hand, boolean lattices form classical categories which are not generally compact, but have trivial hom-sets. The product of any two classical categories is a classical category. In particular, $(\mathbf{Rel}, \times) \times \mathbf{B}$, where $\mathbf{B}$ is a boolean lattice, is a non-compact classical category with non-trivial hom-sets. However, what seems to be lacking is a more natural example of a non-compact classical category with non-trivial hom-sets. Categories of games and strategies seem to be natural candidates. Also, the *double gluing* construction (Loader 1994, Tan 1997, Hyland & Schalk 2003) is known to turn compact closed categories into non-compact

$*$-autonomous categories (i.e., non-compact symmetric linearly distributive categories with negation). It would be interesting to check whether there are circumstances in which the extra structure of a (compact) classical category survives this construction. In other words: can double gluing be extended to classical logic just as we extended GoI to classical logic?

Term calculi and programming It would be interesting to study term calculi for Dummett categories and classical categories.

A classical category is essentially a $*$-autonomous category with symmetric comonoids satisfying certain conditions that result in hom-semilattices. In private communications, Hasegawa has suggested using a modified version of the multiplicative fragment his lambda calculus DCLL (*Dual Classical Linear Logic*) (Hasegawa 2002). To be precise, his approach is based on the lambda calculus below, which is sound and complete with respect to $*$-autonomous categories with symmetric comonoids:

Types

$$\sigma ::= b \mid \bot \mid \sigma \to \sigma$$

Terms

$$\frac{}{\Gamma_1, x : \sigma, \Gamma_2 \vdash x : \sigma}\ (\mathrm{Ax})$$

$$\frac{\Gamma, x : \sigma_1 \vdash M : \sigma_2}{\Gamma \vdash \lambda x^{\sigma_1}.M : \sigma_1 \to \sigma_2}\ (\to I) \qquad \frac{\Gamma \vdash M : \sigma_1 \to \sigma_2 \quad \Gamma \vdash \sigma_2}{\Gamma \vdash MN : \sigma_2}\ (\to E)$$

$$\frac{}{\Gamma \vdash \mathsf{C}_\sigma : ((\sigma \to \bot) \to \bot) \to \sigma}\ (\neg\neg E)$$

Axioms

(β_{lin})	$(\lambda x.E[x])N$	$= E[N]$	
(η)	$\lambda x.Mx$	$= M$	$(x \notin FV(M))$
(C_1)	$L(\mathsf{C}_\sigma M)$	$= ML$	$(L : \sigma \to \bot)$
(C_2)	$\mathsf{C}_\sigma(\lambda k^{\sigma\to\bot}.kM)$	$= M$	$(k \notin FV(M))$
(β_{var})	$(\lambda x.M)y$	$= M[y/x]$	

$E[-]$ stands for a lambda term with a single hole. The laws (C_1) and (C_2) state essentially that C_σ is the left and right inverse of the evident lambda term $\sigma \to ((\sigma \to \bot) \to \bot)$. The first four laws characterize $*$-autonomous categories. The law (β_{var}) allows non-linear substitutions, but only if the arguments are variables. This allows to express the multiplication and unit of the symmetric comonoids:

Derived constructs

$$\begin{aligned}
\top &= \bot \to \bot \\
\sigma_1 \wedge \sigma_2 &= (\sigma_1 \to \sigma_2 \to \bot) \to \bot \\
\sigma_1 \vee \sigma_2 &= (\sigma_1 \to \bot) \to (\sigma_2 \to \bot) \to \bot \\
\langle\rangle_\sigma &= \lambda x^\sigma.\lambda u^\bot.u \\
\blacktriangle_\sigma &= \lambda x^\sigma.\lambda k^{\sigma\to\sigma\to\bot}.kxx \\
&\ \vdots
\end{aligned}$$

As it turns out, the extra axioms required for a classical category can be given as follows:

$$(\sigma) \quad (\lambda x.M)N \preccurlyeq M[N/x]$$

$$\frac{M \preccurlyeq N}{E[M] \preccurlyeq E[N]} \qquad \frac{M \preccurlyeq N \quad N \preccurlyeq M}{M = N}$$

The order $\preccurlyeq$ turns out to be derivable from the hom-semilattice operation

$$\frac{M, N : \sigma \qquad x, k \notin FV(M), FV(N)}{M * N = \mathsf{C}_\sigma(\lambda k^{\sigma \to \bot}.(\lambda x^{\bot}.kM)(kN)) : \sigma}.$$

It should be interesting to deepen the study of classical categories via this lambda calculus.

However, this calculus can only be used for Dummett categories with negation. Also, its syntax hides the beautiful self-duality of the structure. So it is tempting to devise a self-dual, negation-free term calculus for Dummett categories. Such a calculus might be based on the *circuit expressions* in (Blute et al. 1996), on term calculi for the classical sequent calculus along the lines of (Curien & Herbelin 2000, Wadler 2003). Expressions in such calculi can be seen as functional programs with an unspecified evaluation strategy, while MIX introduces an element of parallelism. Lafont's example corresponds to a critical pair which can be resolved by choosing between call-by-value and call-by-name evaluation. In the literature, there seems to be no semantics that models this non-determinism *within one category*. Dummett categories or something similar might help here.

Other starting points for a term-language for classical categories might be Filinski's symmetric lambda calculus (Filinski 1989) and the symmetric lambda calculus by Barbanera and Berardi (Barbanera & Berardi 1996).

Extending Dummett categories to first-order logic Finally, we should like to mention the possibility of extending our categorical semantics to first-order classical logic. This can be achieved using certain indexed categories whose fibres are classical categories. This idea is being explored in Richard McKinley's doctoral work.

Acknowledgments. We are very grateful to Masahito Hasegawa, who contributed many ideas and results in private communications. We should also like to thank Robin Cockett, Martin Hyland, Francois Lamarche, Richard McKinley, Eike Ritter, Phil Scott, Lutz Straßburger, John Power, Edmund Robinson, Robert Seely, and Christian Urban for their comments and criticism.

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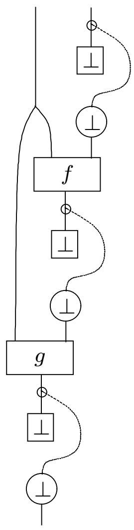
A Some lemmas and proofs

Proof of Lemma 3.9. Applying $Expand_{\perp}$ to the left-hand side of Equation 1 yields



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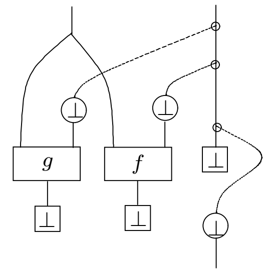
By empire rewiring, we get

$h =$ 

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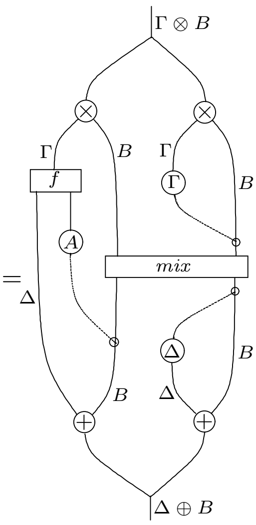
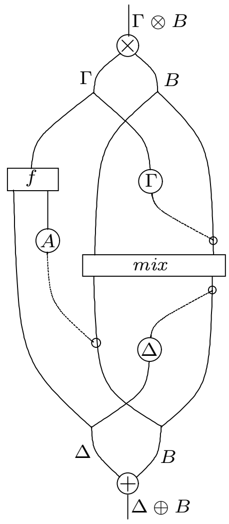
By applying similar transformations to the right-hand side of Equation 1, we also get h. (The fact that f and g appear in opposite order is compensated by the twisted wires in the right-hand side of Equation 1.)

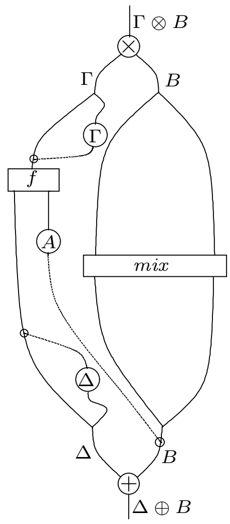
The cases for $n = 0$ and $n \geq 2$ are similar. ☐

Proof of Lemma 3.30. Let l and r be the left-hand side (resp. right-hand side) of *Reduce*WL. We show $l * r = r$. We have

$l * r =$ 

$=$ (by CL-*pointwise* and CR-*pointwise*)


= (by empire rewiring)



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By WL-*neutral* and WR-*neutral*, we can remove the weakenings that introduce Γ and Δ. Because $\blacktriangledown_B \circ mix_{BB} \circ \blacktriangle_B = id_B * id_B = id_B$, we obtain r. □

Lemma A.1. *In every pre-Dummett category, it holds that*

$$(\pi_1^{AB} \otimes \pi_2^{AB}) \circ \blacktriangle_{A\otimes B} = id_{A\otimes B} \quad (8)$$

$$\blacktriangledown_A \circ mix_{AA}^{\langle\rangle} = \pi_1^{AA} * \pi_2^{AA} \quad (9)$$

$$mix_{AB}^{\langle\rangle} = (\iota_1^{AB} \circ \pi_1^{AB}) * (\iota_2^{AB} \circ \pi_2^{AB}) \quad (10)$$

Proof. Equation 8 follows from a routine calculation using the laws ▲*pointwise* and ⟨⟩*neutral*. The see Equation 9, consider

$$\begin{aligned} \blacktriangledown_A \circ mix_{AA}^{\langle\rangle} &= \blacktriangledown_A \circ mix_{AA}^{\langle\rangle} \circ (\pi_1^{AA} \otimes \pi_2^{AA}) \circ \blacktriangle_{A\otimes A} && \text{(by Equation 8)} \\ &= \pi_1^{AA} * \pi_2^{AA} && \text{(by definition of } *) \end{aligned}$$

Equation 10 holds because, by definition of $*$ and the naturality of $mix^{\langle\rangle}$, the morphism $(\iota_1^{AB} \circ \pi_1^{AB}) * (\iota_2^{AB} \circ \pi_2^{AB})$ is equal to

$$\blacktriangledown_{A\oplus B} \circ (\iota_1^{AB} \oplus \iota_2^{AB}) \circ mix_{AB}^{\langle\rangle} \circ (\pi_1^{AB} \otimes \pi_2^{AB}) \circ \blacktriangle_{A\otimes B},$$

which by Equation 8 and its dual is equal to $mix_{AB}^{\langle\rangle}$. □

Lemma A.2. *In every Dummett category it holds that*

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} \leq mix_{AA}.$$

Proof. By Equation 9 and its dual, we have

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} = (\iota_1^{AA} * \iota_2^{AA}) \circ (\pi_1^{AA} * \pi_2^{AA}).$$

Because $*$ is the greatest lower bound with respect to $\leq$, and because $\circ$ is monotonic in both arguments, we have

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} \leq \iota_k^{AA} \circ \pi_k^{AA}$$

for $k \in \{1, 2\}$. So

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} \leq (\iota_1^{AA} \circ \pi_1^{AA}) * (\iota_2^{AA} \circ \pi_2^{AA}).$$

The claim follows because the right-hand side is equal to mix_{AA} by Equation 10. □